Math 261-00
4–9–2004

Review Problems for Test 3

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. (a) Compute the exact value of $\displaystyle\int_0^2 \int_0^{1+\sqrt{2x-x^2}} dy\, dx$.

(b) Compute the exact value of $\displaystyle\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^4 dz\, dy\, dx$.

2. Find the volume of the solid lying below the paraboloid $z = x^2 + y^2$ and above the region in the x-y plane bounded by $y = x^2$ and $y = x + 2$.

3. (a) Compute $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \frac{1}{\sqrt{x^3+4}}\, dx\, dy$.

(b) Compute

$$\int_0^2 \int_{y/2}^1 3y^2 \cos(x^4)\, dx\, dy + \int_{-2}^0 \int_{-y/2}^1 3y^2 \cos(x^4)\, dx\, dy.$$

4. Compute $\displaystyle\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$.

5. Find the volume of the region which lies above the cone $z = \dfrac{1}{\sqrt{3}}\sqrt{x^2+y^2}$ and below the hemisphere $z = \sqrt{1-x^2-y^2}$.

6. Compute $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{x+y} \sqrt{x^2+y^2}\, dz\, dy\, dx$.

7. Find the area of the surface

$$x = 2u\cos v, \quad y = u^2, \quad z = 2u \sin v, \quad 0 \le u \le 1, \quad 0 \le v \le 2\pi.$$

8. Find the center of mass of the region in the first octant cut off by the plane $2x + 2y + z = 4$, if the density is $\rho(x, y, z) = 2z + 1$.

9. Compute $\displaystyle\iint_R (x - 2y)\, dx\, dy$, where R is the parallelogram bounded by $y = x+1$, $y = x-2$, $y = -\dfrac{1}{2}x + 1$, and $y = -\dfrac{1}{2}x + 4$.

10. (a) Compute $\displaystyle\int_{\vec{\sigma}} (x+y)\, dx - (x-y)\, dy$, where σ is the path consisting of the segment from $(0,1)$ to $(-1,0)$, the segment from $(-1,0)$ to $(0,-1)$, the segment from $(0,-1)$ to $(1,0)$, and the segment from $(1,0)$ to $(0,1)$.

(b) Compute $\displaystyle\int_{\vec{\sigma}} \vec{F} \cdot d\vec{s}$, where $\vec{\sigma}$ is the curve of intersection of $x^2 + y^2 = 1$ and the plane $z = 2 + 2x + 3y$, traversed counterclockwise as viewed from above, and $\vec{F} = \langle -2y, 2x, 2 \rangle$.

11. Let $\vec{F}(x,y,z) = \langle x^2 y + z, xz, x + 3yz \rangle$. Compute $\operatorname{curl} \vec{F}$ and $\operatorname{div} \vec{F}$.

12. Compute

$$\int_{\vec{\sigma}} (y^2 + z^3)\, dx + (2xy - 2y)\, dy + (3xz^2 + 4)\, dz,$$

where $\vec{\sigma}(t)$ is the path which consists of the curve $\left\langle \dfrac{3t}{2t+1}, te^{2(t-1)}, \dfrac{1}{8}t^2(t^2+1)^3 \right\rangle$ for $0 \le t \le 1$, followed by the segment from $(1, 1, 1)$ to $(1, 2, -1)$.

13. Compute $\displaystyle\int_{\vec{\sigma}} (x^2 y - xy^2)\, dx + \left(2x^2 y + \frac{1}{3}x^3\right) dy$, where $\vec{\sigma}$ is the boundary of the square $0 \le x \le 1$, $0 \le y \le 1$, traversed in the counterclockwise direction.

14. Find the area of the region enclosed by the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Solutions to the Review Problems for Test 3

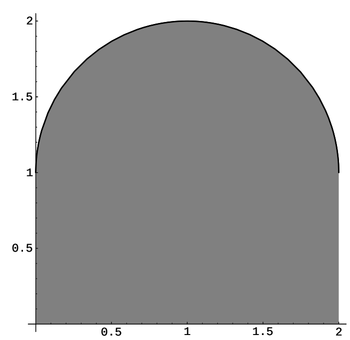
1. (a) Compute the exact value of $\displaystyle\int_0^2 \int_0^{1+\sqrt{2x-x^2}} dy\, dx$.

Rewrite $y = 1 + \sqrt{2x - x^2}$:

$$y - 1 = \sqrt{2x - x^2}, \quad (y-1)^2 = 2x - x^2, \quad x^2 - 2x + (y-1)^2 = 0, \quad x^2 - 2x + 1 + (y-1)^2 = 1,$$

$$(x-1)^2 + (y-1)^2 = 1.$$

Thus, $y = 1 + \sqrt{2x - x^2}$ is the top half of the circle of radius 1 centered at $(1, 1)$. The region is bounded above by this semicircle, below by the x-axis, and on the sides by $x = 0$ and $x = 2$:



Since the integrand is 1, the integral represents the area of the region. The area is the sum of the area of the semicircle (which has radius 1) and the rectangle below it (which is 2 by 1). Thus,

$$\int_0^2 \int_0^{1+\sqrt{2x-x^2}} dy\, dx = \frac{1}{2}\pi \cdot 1^2 + 2 \cdot 1 = 2 + \frac{\pi}{2}. \quad \square$$

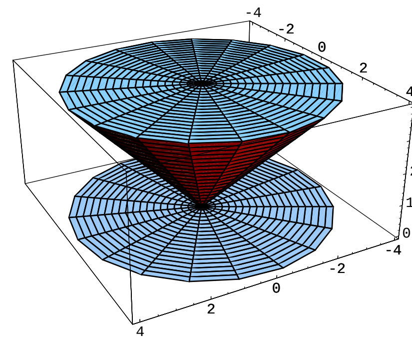
(b) Compute the exact value of $\displaystyle\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^{4} dz\, dy\, dx$.

The projection of the region into the x-y plane is

$$\begin{array}{c} -4 \le x \le 4 \\ -\sqrt{16-x^2} \le y \le \sqrt{16-x^2} \end{array}.$$

This is the circle of radius 4 centered at the origin.

The bottom of the region is the cone $z = \sqrt{x^2 + y^2}$ and the top is the plane $z = 4$.



Thus, the region is a cone with height $h = 4$ and radius $r = 4$.

Since the integrand is 1, the integral represent the volume of the region. A cone of height h and radius r has volume $\dfrac{1}{3}\pi r^2 h$. Therefore,

$$\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^{4} dz\, dy\, dx = \frac{1}{3}\pi \cdot 4^2 \cdot 4 = \frac{64\pi}{3}. \quad \square$$

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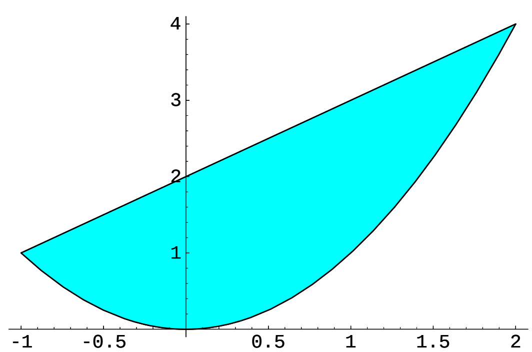
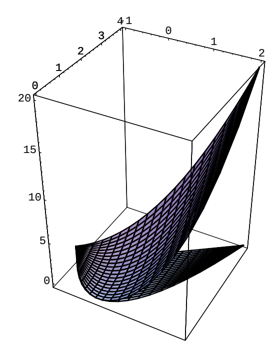
2. Find the volume of the solid lying below the paraboloid $z = x^2 + y^2$ and above the region in the x-y plane bounded by $y = x^2$ and $y = x + 2$.

The projection into the x-y plane is shown in the first picture:





Since x^2 and $x+2$ intersect at $x = -1$ and at $x = 2$, the region is described by the following inequalities:

$$-1 \le x \le 2, \quad x^2 \le y \le x + 2.$$

The top of the solid is $z = x^2 + y^2$. The bottom is the x-y plane $z = 0$. The second picture shows the top and the bottom; the solid is the region between them.

The volume is

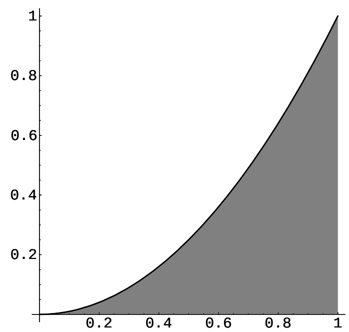
$$\int_{-1}^{2} \int_{x^2}^{x+2} (x^2 + y^2)\, dy\, dx = \int_{-1}^{2} \left[x^2 y + \frac{1}{3} y^3 \right]_{x^2}^{x+2} dx = \int_{-1}^{2} \left(x^3 + 2x^2 + \frac{1}{3}(x+2)^3 - x^5 - \frac{1}{3}x^6 \right) dx =$$

$$\left[\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{12}(x+2)^4 - \frac{1}{6}x^6 - \frac{1}{21}x^7 \right]_{-1}^{2} = \frac{639}{35} \approx 18.25714. \quad \square$$

3. (a) Compute $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \frac{1}{\sqrt{x^3+4}}\,dx\,dy$.

Interchange the order of integration:

$$\left\{\begin{array}{c} 0 \le y \le 1 \\ \sqrt{y} \le x \le 1 \end{array}\right\} \quad \to$$



$$\to \quad \left\{\begin{array}{c} 0 \le x \le 1 \\ 0 \le y \le x^2 \end{array}\right\}$$

Thus,

$$\int_0^1 \int_{\sqrt{y}}^1 \frac{1}{\sqrt{x^3+4}}\,dx\,dy = \int_0^1 \int_0^{x^2} \frac{1}{\sqrt{x^3+4}}\,dy\,dx = \int_0^1 \frac{1}{\sqrt{x^3+4}}\,[y]_0^{x^2}\,dx = \int_0^1 \frac{x^2}{\sqrt{x^3+4}}\,dx =$$

$$\left[\frac{2}{3}(x^3+4)^{1/2}\right]_0^1 = \frac{2}{3}(\sqrt{5}-2) \approx 0.15738.$$

Here's the work for the integral:

$$\int \frac{x^2}{\sqrt{x^3+4}}\,dx = \int \frac{x^2}{\sqrt{u}} \cdot \frac{du}{3x^2} = \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{2}{3}\sqrt{u} + c = \frac{2}{3}\sqrt{x^3+4} + c.$$

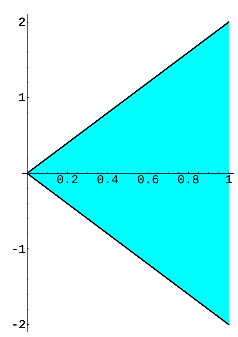
$$\left[u = x^3 + 4, \quad du = 3x^2\,dx, \quad dx = \frac{du}{3x^2}\right] \quad \square$$

(b) Compute

$$\int_0^2 \int_{y/2}^1 3y^2\cos(x^4)\,dx\,dy + \int_{-2}^0 \int_{-y/2}^1 3y^2\cos(x^4)\,dx\,dy.$$

Interchange the order of integration:

$$\left\{\begin{array}{c} 0 \le y \le 2 \\ \dfrac{y}{2} \le x \le 1 \end{array}\right\} \quad \left\{\begin{array}{c} -2 \le y \le 0 \\ -\dfrac{y}{2} \le x \le 1 \end{array}\right\} \quad \to$$



$$\to \quad \left\{\begin{array}{c} 0 \le x \le 1 \\ -2x \le y \le 2x \end{array}\right\}$$

Thus,

$$\int_0^2 \int_{y/2}^1 3y^2\cos(x^4)\,dx\,dy + \int_{-2}^0 \int_{-y/2}^1 3y^2\cos(x^4)\,dx\,dy = \int_0^1 \int_{-2x}^{2x} 3y^2\cos(x^4)\,dy\,dx =$$

$$\int_0^1 \cos(x^4)\left[y^3\right]_{-2x}^{2x} dx = 16\int_0^1 x^3\cos(x^4)\,dx = 4\left[\sin(x^4)\right]_0^1 = 4\sin 1 \approx 3.36588.$$

Here's the work for the integral:

$$\int x^3\cos(x^4)\,dx = \int x^3\cos u\cdot\frac{du}{4x^3} = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + c = \frac{1}{4}\sin(x^4)+c.$$

$$\left[u = x^4, \quad du = 4x^3\,dx, \quad dx = \frac{du}{4x^3}\right] \quad \Box$$

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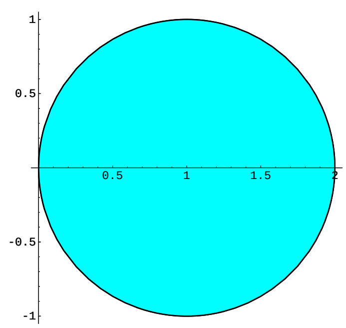
4. Compute $\displaystyle\int_0^2\int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$.

Note that $y = \pm\sqrt{2x-x^2}$ may be rewritten as follows:

$$y^2 = 2x - x^2, \quad x^2 - 2x + y^2 = 0, \quad x^2 - 2x + 1 + y^2 = 1, \quad (x-1)^2 + y^2 = 1.$$

This is a circle of radius 1 centered at $(1,0)$.
I'll convert to polar:

$$\left\{\begin{array}{c} 0 \le x \le 2 \\ -\sqrt{2x-x^2} \le y \le \sqrt{2x-x^2} \end{array}\right\} \quad \to$$



$$\to \quad \left\{\begin{array}{c} -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2} \\ 0 \le r \le 2\cos\theta \end{array}\right\}$$

To get the polar equation for the circle, start with $x^2 - 2x + y^2 = 0$. Then

$$x^2 + y^2 = 2x, \quad r^2 = 2r\cos\theta, \quad r = 2\cos\theta.$$

Note that the whole circle is traced out *once* as θ goes from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$ (not, for example, from 0 to 2π).
The integrand is $\sqrt{x^2+y^2} = \sqrt{r^2} = r$. So

$$\int_0^2\int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{1}{3}r^3\right]_0^{2\cos\theta} d\theta =$$

$$\frac{8}{3}\int_{-\pi/2}^{\pi/2}(\cos\theta)^3\,d\theta = \frac{8}{3}\left[\sin\theta - \frac{1}{3}(\sin\theta)^3\right]_{-\pi/2}^{\pi/2} = \frac{32}{9}.$$

Here's the work for the integral:

$$\int(\cos\theta)^3\,d\theta = \int(\cos\theta)^2(\cos\theta)\,d\theta = \int\left(1-(\sin\theta)^2\right)(\cos\theta)\,d\theta = \int(1-u^2)\cos\theta\cdot\frac{du}{\cos\theta} = \int(1-u^2)\,du =$$

$$\left[u = \sin\theta, \quad du = \cos\theta\,d\theta, \quad d\theta = \frac{du}{\cos\theta}\right]$$

$$u - \frac{1}{3}u^3 + c = \sin\theta - \frac{1}{3}(\sin\theta)^3 + c. \quad \square$$

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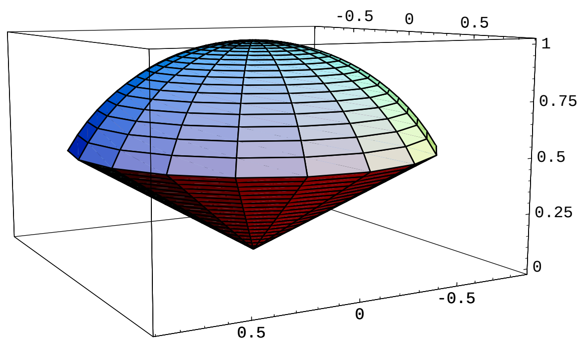
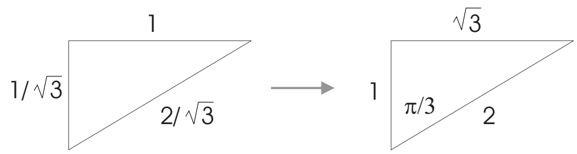
5. Find the volume of the region which lies above the cone $z = \dfrac{1}{\sqrt{3}}\sqrt{x^2+y^2}$ and below the hemisphere $z = \sqrt{1-x^2-y^2}$.



I'll do the integral in spherical coordinates. It's pretty clear that the ranges for θ and ρ are $0 \le \theta \le 2\pi$ and $0 \le \rho \le 1$. What is the range for ϕ? I need to figure out the angle between the side of the cone and the z-axis.

To do this, take a random point on the cone: For instance, if $x = 1$ and $y = 0$, then $z = \dfrac{1}{\sqrt{3}}$. Here's the picture:



I drew a triangle with horizontal side 1 (since $r = \sqrt{x^2+y^2} = \sqrt{1^2+0^2} = 1$) and vertical side $\dfrac{1}{\sqrt{3}}$ (the value of z). I found the hypotenuse using Pythagoras. Then I scaled the triangle up by multiplying all the sides by $\sqrt{3}$ so I could see the ratios better. In the second triangle, I can clearly see that the cone angle is $\dfrac{\pi}{3}$.

Therefore, the range on ϕ is $0 \le \phi \le \dfrac{\pi}{3}$.

The volume is

$$\iiint_R dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\pi/3} \sin\phi\left[\frac{1}{3}\rho^3\right]_0^1 d\phi = \frac{2\pi}{3}\int_0^{\pi/3}\sin\phi\,d\phi =$$

$$\frac{2\pi}{3}\left[-\cos\phi\right]_0^{\pi/3} = \frac{\pi}{3} \approx 1.04720. \quad \square$$

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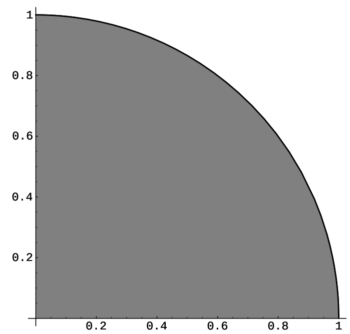
6. Compute $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{x+y}\sqrt{x^2+y^2}\,dz\,dy\,dx$.

I'll convert to cylindrical coordinates. The ranges $0 \le x \le 1$ and $0 \le y \le \sqrt{1-x^2}$ describe the interior of the circle of radius 1 centered at the origin which lies in the first quadrant:



In polar coordinates, it is

$$\begin{array}{c} 0 \le \theta \le \dfrac{\pi}{2} \\ 0 \le r \le 1 \end{array}.$$

The limits on z become $0 \le z \le r\cos\theta + r\sin\theta$.
The integrand is $\sqrt{x^2+y^2} = \sqrt{r^2} = r$.
Therefore,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{x+y} \sqrt{x^2+y^2}\, dz\, dy\, dx = \int_0^{\pi/2} \int_0^1 \int_0^{r\cos\theta + r\sin\theta} r^2\, dz\, dr\, d\theta =$$

$$\int_0^{\pi/2} \int_0^1 r^2 \left[z\right]_0^{r\cos\theta + r\sin\theta} dr\, d\theta = \int_0^{\pi/2} (\cos\theta + \sin\theta) \int_0^1 r^3\, dr\, d\theta = \int_0^{\pi/2} (\cos\theta + \sin\theta) \left[\frac{1}{4} r^4\right]_0^1 d\theta =$$

$$\frac{1}{4} \int_0^{\pi/2} (\cos\theta + \sin\theta)\, d\theta = \frac{1}{4} \left[\sin\theta - \cos\theta\right]_0^{\pi/2} = \frac{1}{2}. \quad \square$$

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7. Find the area of the surface

$$x = 2u\cos v, \quad y = u^2, \quad z = 2u\sin v, \quad 0 \le u \le 1, \quad 0 \le v \le 2\pi.$$

$$\vec{T}_u = \langle 2\cos v, 2u, 2\sin v\rangle, \quad \vec{T}_v = \langle -2u\sin v, 0, 2u\cos v\rangle,$$

$$\vec{T}_u \times \vec{T}_v = \left|\begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2\cos v & 2u & 2\sin v \\ -2u\sin v & 0 & 2u\cos v \end{array}\right| = \langle 4u^2\cos v, -4u(\sin v)^2 - 4u(\cos v)^2, 4u^2\sin v\rangle =$$

$$\langle 4u^2\cos v, -4u, 4u^2\sin v\rangle,$$

$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{16u^4(\cos v)^2 + 16u^2 + 16u^4(\sin v)^2} = \sqrt{16u^4 + 16u^2} = 4u\sqrt{u^2+1}.$$

The area is

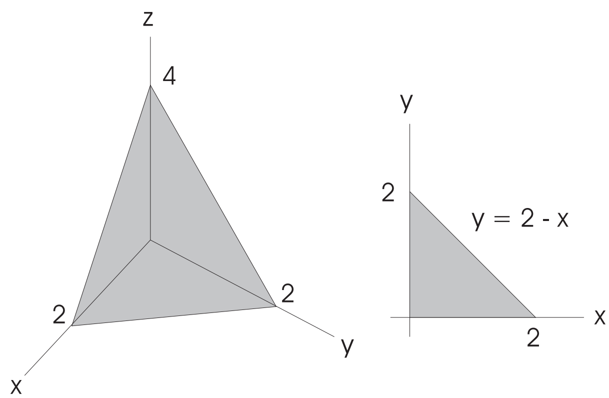
$$\int_0^{2\pi} \int_0^1 4u\sqrt{u^2+1}\, du\, dv = 8\pi \int_0^1 u\sqrt{u^2+1}\, du = 8\pi \left[\frac{1}{3}(u^2+1)^{3/2}\right]_0^1 = \frac{8\pi}{3}(2^{3/2} - 1) \approx 15.31780.$$

Here's the work for the integral:

$$\int u\sqrt{u^2+1}\, du = \int u\sqrt{w} \cdot \frac{dw}{2u} = \frac{1}{2} \int \sqrt{w}\, dw = \frac{1}{3} w^{3/2} + c = \frac{1}{3}(u^2+1)^{3/2} + c.$$

$$\left[w = u^2 + 1, \quad dw = 2u\,du, \quad du = \frac{dw}{2u}\right] \quad \square$$

8. Find the center of mass of the region in the first octant cut off by the plane $2x+2y+z=4$, if the density is $\rho(x,y,z) = 2z+1$.



The region is shown in the first picture. The top is the plane $z = 4-2x-2y$, and the bottom is $z=0$. Thus, the range for z is $0 \le z \le 4-2x-2y$.

The projection into the x-y plane is shown in the second picture. It is

$$\begin{array}{c} 0 \le x \le 2 \\ 0 \le y \le 2-x \end{array}.$$

The mass is

$$\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (2z+1)\,dz\,dy\,dx = \int_0^2 \int_0^{2-x} \left[z^2+z\right]_0^{4-2x-2y} dy\,dx =$$

$$\int_0^2 \int_0^{2-x} \left((4-2x-2y)^2 + (4-2x-2y)\right) dy\,dx = \int_0^2 \left[-\frac{1}{6}(4-2x-2y)^3 - \frac{1}{4}(4-2x-2y)^2\right]_0^{2-x} dx =$$

$$\int_0^2 \left(\frac{1}{6}(4-2x)^3 + \frac{1}{4}(4-2x)^2\right) dx = \left[-\frac{1}{48}(4-2x)^4 - \frac{1}{24}(4-2x)^3\right]_0^2 = 8.$$

The region is symmetric and x and y, and so is the density function. Therefore, $\overline{x} = \overline{y}$.

I'm going to omit the ugly details of the integrations for the moments.

The x-moment is

$$\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} x(2z+1)\,dz\,dy\,dx = \frac{52}{15}.$$

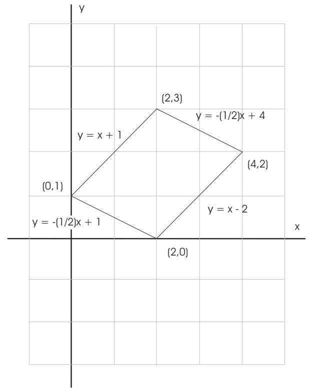
Hence, $\overline{x} = \dfrac{\frac{52}{15}}{8} = \dfrac{13}{30}$. Likewise, $\overline{y} = \dfrac{13}{30}$.

The z-moment is

$$\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} z(2z+1)\,dz\,dy\,dx = \frac{56}{5}.$$

Hence, $\overline{z} = \dfrac{\frac{56}{5}}{8} = \dfrac{7}{5}$. $\square$

9. Compute $\displaystyle\iint_R (x-2y)\,dx\,dy$, where R is the parallelogram bounded by $y=x+1$, $y=x-2$, $y=-\dfrac{1}{2}x+1$, and $y=-\dfrac{1}{2}x+4$.



I graphed the lines and found the intersection points. Next, I'll construct a transformation from the square $0 \le u \le 1$, $0 \le v \le 1$, onto the parallelogram. I'll use $(0,1)$ as my reference point.

The vector from $(0,1)$ to $(2,0)$ is $\langle 2,-1\rangle$. The vector from $(0,1)$ to $(2,3)$ is $\langle 2,2\rangle$. The transformation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If I multiply out and combine terms on the right, then equate corresponding components, I get

$$x = 2u+2v, \quad y = -u+2v+1, \quad 0 \le u \le 1, \quad 0 \le v \le 1.$$

The Jacobian is

$$\left|\det \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\right| = 6.$$

The integrand is

$$x - 2y = (2u+2v) - 2(-u+2v+1) = 4u - 2v - 2.$$

Hence,

$$\iint_R (x-2y)\,dx\,dy = \int_0^1 \int_0^1 (4u-2v-2)(6)\,du\,dv = 6\int_0^1 \left[2u^2 - 2uv - 2u\right]_0^1 dv = 6\int_0^1 (-2v)\,dv =$$

$$-12\left[\frac{1}{2}v^2\right]_0^1 = -6. \quad \square$$

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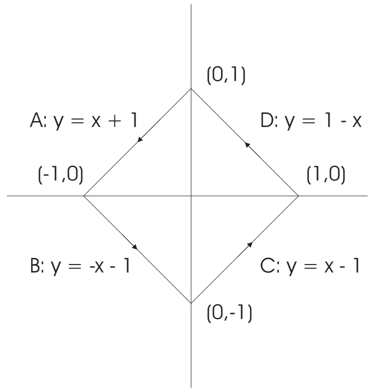
10. (a) Compute $\displaystyle\int_{\vec{\sigma}} (x+y)\,dx - (x-y)\,dy$, where σ is the path consisting of the segment from $(0,1)$ to $(-1,0)$, the segment from $(-1,0)$ to $(0,-1)$, the segment from $(0,-1)$ to $(1,0)$, and the segment from $(1,0)$

to $(0, 1)$.



I'll break the integral up into four segments, as shown in the picture.

Segment A is $y = x + 1$. $x + y = 2x + 1$, $x - y = -1$, $dy = dx$, and x goes from 0 to -1. The integral is

$$\int_0^{-1} ((2x + 1)\,dx - (-1)\,dx) = \int_0^{-1} (2x + 2)\,dx = \left[x^2 + 2x\right]_0^{-1} = -1.$$

Segment B is $y = -x - 1$. $x + y = -1$, $x - y = 2x + 1$, $dy = -dx$, and x goes from -1 to 0. The integral is

$$\int_{-1}^{0} (-dx - (2x + 1)\,dx) = \int_{-1}^{0} (-2x - 2)\,dx = \left[-x^2 - 2x\right]_{-1}^{0} = -1.$$

Segment C is $y = x - 1$. $x + y = 2x - 1$, $x - y = 1$, $dy = dx$, and x goes from 0 to 1. The integral is

$$\int_0^1 ((2x - 1)\,dx - dx) = \int_0^1 (2x - 2)\,dx = \left[x^2 - 2x\right]_0^1 = -1.$$

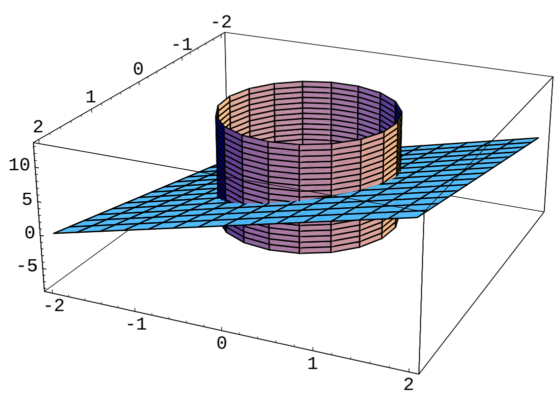
Segment D is $y = 1 - x$. $x + y = 1$, $x - y = 2x - 1$, $dy = -dx$, and x goes from 1 to 0. The integral is

$$\int_1^0 (dx - (2x - 1)dx) = \int_1^0 (2 - 2x)\,dx = \left[2x - x^2\right]_1^0 = -1.$$

Therefore,

$$\int_{\vec{\sigma}} (x + y)\,dx - (x - y)\,dy = -1 + (-1) + (-1) + (-1) = -4. \quad \square$$

(b) Compute $\displaystyle\int_{\vec{\sigma}} \vec{F} \cdot d\vec{s}$, where $\vec{\sigma}$ is the curve of intersection of $x^2 + y^2 = 1$ and the plane $z = 2 + 2x + 3y$, traversed counterclockwise as viewed from above, and $\vec{F} = \langle -2y, 2x, 2 \rangle$.



The projection of the curve into the x-y plane is the circle $x^2 + y^2 = 1$, which may be parametrized by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Note that this parameter range traverses the circle counterclockwise as viewed from above.

Plugging these expressions into $z = 2 + 2x + 3y$, I get $z = 2 + 2\cos t + 3\sin t$. Hence, the curve of intersection is

$$\vec{\sigma}(t) = \langle \cos t, \sin t, 2 + 2\cos t + 3\sin t \rangle.$$

Therefore,

$$\vec{\sigma}\,'(t) = \langle -\sin t, \cos t, -2\sin t + 3\cos t \rangle.$$

The integrand is

$$\vec{F}(\vec{\sigma}(t)) \cdot \vec{\sigma}\,'(t) = \langle -2\sin t, 2\cos t, 2 \rangle \cdot \langle -\sin t, \cos t, -2\sin t + 3\cos t \rangle =$$

$$2(\sin t)^2 + 2(\cos t)^2 - 4\sin t + 6\cos t = 2 - 4\sin t + 6\cos t.$$

The integral is

$$\int_{\vec{\sigma}} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (2 - 4\sin t + 6\cos t)\ dt = \left[2t + 4\cos t + 6\sin t\right]_0^{2\pi} = 4\pi \approx 12.56637. \quad \square$$

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11. Let $\vec{F}(x, y, z) = \langle x^2y + z, xz, x + 3yz \rangle$. Compute $\operatorname{curl} \vec{F}$ and $\operatorname{div} \vec{F}$.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{d}{dz} \\ x^2y + z & xz & x + 3yz \end{vmatrix} = \langle 3z - x, 1 - 1, z - x^2 \rangle = \langle 3z - x, 0, z - x^2 \rangle.$$

$$\operatorname{div} \vec{F} = 2xy + 0 + 3y = 2xy + 3y. \quad \square$$

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12. Compute

$$\int_{\vec{\sigma}} (y^2 + z^3)\, dx + (2xy - 2y)\, dy + (3xz^2 + 4)\, dz,$$

where $\vec{\sigma}(t)$ is the path which consists of the curve $\left\langle \dfrac{3t}{2t+1}, te^{2(t-1)}, \dfrac{1}{8}t^2(t^2+1)^3 \right\rangle$ for $0 \le t \le 1$, followed by the segment from $(1, 1, 1)$ to $(1, 2, -1)$.

It would be very tedious to compute the line integral directly, and it should lead you to ask yourself whether there might not be an easier way. Well,

$$\operatorname{curl}\langle y^2 + z^3, 2xy - 2y, 3xz^2 + 4 \rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 + z^3 & 2xy - 2y & 3xz^2 + 4 \end{vmatrix} = \langle 0, 3z^2 - 3z^2, 2y - 2y \rangle = \langle 0, 0, 0 \rangle.$$

The field is conservative. I'll find a potential function f. I want

$$\frac{\partial f}{\partial x} = y^2 + z^3, \quad \frac{\partial f}{\partial y} = 2xy - 2y, \quad \frac{\partial f}{\partial z} = 3xz^2 + 4.$$

Integrate the first equation with respect to x:

$$f(x, y, z) = \int (y^2 + z^3)\, dx = xy^2 + xz^3 + C(y, z).$$

$C(y,z)$ is an arbitrary constant depending on y and z. Differentiate with respect to y and set the result equal to $\dfrac{\partial f}{\partial y} = 2xy - 2y$:

$$2xy + \frac{\partial C}{\partial y} = \frac{\partial f}{\partial y} = 2xy - 2y.$$

Cancelling $2xy$'s, I get $\dfrac{\partial C}{\partial y} = -2y$, so

$$C = \int -2y\,dy = -y^2 + D(z).$$

$D(z)$ is an arbitrary constant depending on z. Then

$$f(x,y,z) = xy^2 + xz^3 - y^2 + D(z).$$

Differentiate with respect to z and set the result equal to $\dfrac{\partial f}{\partial z} = 3xz^2 + 4$:

$$3xz^2 + D'(z) = \frac{\partial f}{\partial z} = 3xz^2 + 4.$$

Cancelling $3xz^2$'s, I get $D'(z) = 4$, so $D(z) = 4z + E$. Now E is a numerical arbitrary constant, and since I need *some* potential function, I can take $E = 0$. Then

$$f(x,y,z) = xy^2 + xz^3 - y^2 + 4z.$$

Now $\vec{\sigma}$ starts at $(0,0,0)$ (as you see by plugging $t = 0$ into $\left\langle \dfrac{3t}{2t+1}, te^{2(t-1)}, \dfrac{1}{8}t^2(t^2+1)^3 \right\rangle$), and it ends at $(1,2,-1)$. By path independence,

$$\int_{\vec{\sigma}} (y^2 + z^3)\,dx + (2xy - 2y)\,dy + (3xz^2 + 4)\,dz = f(1,2,-1) - f(0,0,0) = -5. \quad \square$$

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13. Compute $\displaystyle\int_{\vec{\sigma}} (x^2y - xy^2)\,dx + \left(2x^2y + \frac{1}{3}x^3\right)\,dy$, where $\vec{\sigma}$ is the boundary of the square $0 \le x \le 1$, $0 \le y \le 1$, traversed in the counterclockwise direction.

By Grren's Theorem,

$$\int_{\vec{\sigma}} (x^2y - xy^2)\,dx + \left(2x^2y + \frac{1}{3}x^3\right)\,dy = \int_0^1 \int_0^1 \big((4xy + x^2) - (x^2 - 2xy)\big)\,dx\,dy = \int_0^1 \int_0^1 6xy\,dx\,dy =$$

$$\int_0^1 \left[3x^2y\right]_0^1\,dy = \int_0^1 3y\,dy = \left[\frac{3}{2}y^2\right]_0^1 = \frac{3}{2}. \quad \square$$

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14. Find the area of the region enclosed by the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Parametrize the ellipse by

$$\vec{\sigma}(t) = \langle 2\cos t, 3\sin t\rangle, \quad 0 \le t \le 2\pi.$$

This curve traverses the ellipse counterclockwise. By Green's Theorem, the area is

$$\int_{\vec{\sigma}} x\,dy = \int_0^{2\pi} x \cdot \frac{dy}{dt}\,dt = \int_0^{2\pi} (2\cos t)(3\cos t)\,dy = 6\int_0^{2\pi} (\cos t)^2\,dt = 3\int_0^{2\pi} (1 + \cos 2t)\,dt =$$

$$3\left[t + \frac{1}{2}\sin 2t\right]_0^{2\pi} = 6\pi. \quad \square$$

Soon you will have forgotten the world, and the world will have forgotten you. - Marcus Aurelius