Math 261-00
2–27–2004

Review Sheet for Test 2

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. Find the unit tangent and unit normal for the curve $y = \frac{1}{3}x^3 + x^2 + 3x + \frac{2}{3}$ at the point $(1, 5)$.

2. For the curve

$$\vec{r}(t) = \left\langle \frac{1}{3}t^3 + 1, t^2 + 1, 2t + 5 \right\rangle,$$

find the unit tangent, the unit normal, the binormal, and the osculating circle at $t = 1$.

3. Find the domain and range of $f(x, y, z) = \dfrac{z^2 + 1}{\sqrt{1 - x^2 - y^2}}$.

4. (a) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^4 + 5y^4}{x^4 + 3x^2y^2 + y^4}$ is undefined.

(b) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{x^4 + 3x^2y^2 + y^4}$ is defined and find its value.

5. For what points (x, y) is the function $f(x, y) = \ln(xy)$ continuous?

6. Find the tangent plane to the surface

$$x = u^2 - 3v^2, \quad y = \frac{4u}{v}, \quad z = 2u^2v^3$$

for $u = 1$ and $v = 1$.

7. Use a linear approximation to $z = f(x, y) = x^2 - y^2$ at the point $(2, 1)$ to approximate $f(1.9, 1.1)$.

8. Find the gradient of $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2 + 1}}$ and show that it always points toward the origin.

9. Find the rate of change of $f(x, y, z) = xy - yz + xz$ at the point $(1, -2, -2)$ in the direction toward the origin. Is f increasing or decreasing in this direction?

10. Calvin Butterball sits in his go-cart on the surface

$$z = x^3 - 2x^2y + x^2 + xy^2 - 2y^3 + y^2$$

at the point $(1, 1, 0)$. If his go-cart is pointed in the direction of the vector $\vec{v} = \langle 15, -8\rangle$, at what rate will it roll downhill?

11. Find the tangent plane to $x^2 - y^2 + 2yz + z^5 = 6$ at the point $(2, 1, 1)$.

12. The rate of change of $f(x, y)$ at $(1, -1)$ is 2 in the direction *toward* $(5, -1)$ and is $\frac{6}{5}$ in the direction of the vector $\langle -3, -4\rangle$. Find $\nabla f(1, -1)$.

13. Let r and θ be the standard polar coordinates variables. Use the Chain Rule to find $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$, for $f(x, y) = xe^x + e^y$.

14. Suppose $u = f(x,y,z)$ and $x = \phi(s,t)$, $y = \psi(s,t)$, $z = \mu(s,t)$. Use the Chain Rule to write down an expression for $\dfrac{\partial u}{\partial t}$.

15. Suppose that $w = f(x,y)$, $x = g(r,s,t)$, and $y = h(r,t,s)$. Use the Chain Rule to find an expression for $\dfrac{\partial^2 f}{\partial t^2}$.

16. Locate and classify the critical points of

$$z = x^2 y - 4xy + \frac{1}{3}y^3 - \frac{3}{2}y^2.$$

17. Find the points on the sphere $x^2+y^2+z^2 = 36$ which are closest to and farthest from the point $(4,-3,12)$.

Solutions to the Review Sheet for Test 2

1. Find the unit tangent and unit normal for the curve $y = \dfrac{1}{3}x^3 + x^2 + 3x + \dfrac{2}{3}$ at the point $(1,5)$.

The curve may be parametrized by

$$\vec{r}(t) = \left\langle t, \frac{1}{3}t^3 + t^2 + 3t + \frac{2}{3}\right\rangle.$$

Thus,

$$\vec{r}\,'(t) = \langle 1, t^2 + 2t + 3\rangle, \quad \vec{r}\,'(1) = \langle 1, 6\rangle, \quad \|\vec{r}\,'(1)\| = \sqrt{37}.$$

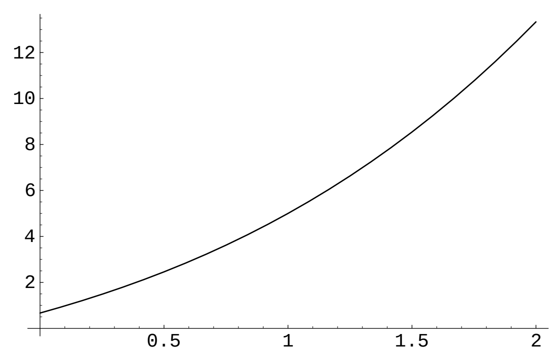
The unit tangent is

$$\vec{T}(1) = \frac{1}{\sqrt{37}}\langle 1, 6\rangle.$$

For a plane curve, I can use geometry to find the unit normal. By swapping components and negating one of them, I can see that the following unit vectors are perpendicular to $\vec{T}(1)$:

$$\frac{1}{\sqrt{37}}\langle -6, 1\rangle, \quad \frac{1}{\sqrt{37}}\langle 6, -1\rangle.$$

Graph the curve near $x = 1$:



From the graph, I can see that the unit normal at $x = 1$ must point up and to the left. This means that the x-component must be negative and the y-component must be positive. Hence,

$$\vec{N}(1) = \frac{1}{\sqrt{37}}\langle -6, 1\rangle.$$

Note that you *can't* use this trick in 3 dimensions, since there are infinitely many vectors perpendicular to the unit tangent. □

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2. For the curve

$$\vec{r}(t) = \left\langle \frac{1}{3}t^3 + 1, t^2 + 1, 2t + 5 \right\rangle,$$

find the unit tangent, the unit normal, the binormal, and the osculating circle at $t = 1$.

$$\vec{r}\,'(t) = \langle t^2, 2t, 2\rangle, \quad \vec{r}\,'(1) = \langle 1, 2, 2\rangle, \quad \|\vec{r}\,'(1)\| = 3.$$

The unit tangent at $t = 1$ is

$$\vec{T}(1) = \frac{1}{3}\langle 1, 2, 2\rangle.$$

Now

$$\|\vec{r}\,'(t)\| = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2+2)^2} = t^2 + 2,$$

so

$$\vec{T}(t) = \left\langle \frac{t^2}{t^2+2}, \frac{2t}{t^2+2}, \frac{2}{t^2+2} \right\rangle.$$

Hence,

$$\vec{T}\,'(t) = \left\langle \frac{4t}{(t^2+2)^2}, \frac{4-2t^2}{(t^2+2)^2}, -\frac{4t}{(t^2+2)^2} \right\rangle,$$

$$\vec{T}\,'(1) = \left\langle \frac{4}{9}, \frac{2}{9}, -\frac{4}{9} \right\rangle = \frac{2}{9}\langle 2, 1, -2\rangle,$$

$$\|\vec{T}\,'(1)\| = \frac{2}{9}\sqrt{2^2 + 1^2 + (-2)^2} = \frac{2}{3}.$$

The unit normal at $t = 1$ is

$$\vec{N}(1) = \frac{1}{\frac{2}{3}} \frac{2}{9}\langle 2, 1, -2\rangle = \frac{1}{3}\langle 2, 1, -2\rangle.$$

The binormal at $t = 1$ is

$$\vec{T}(1) \times \vec{N}(1) = \frac{1}{9}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{vmatrix} = \frac{1}{9}\langle -6, 6, -3\rangle = \frac{1}{3}\langle -2, 2, -1\rangle.$$

Next, I'll compute the curvature.

$$\vec{r}\,'(t) = \langle t^2, 2t, 2\rangle, \quad \text{so} \quad \vec{r}\,''(t) = \langle 2t, 2, 0\rangle, \quad \text{and} \quad \vec{r}\,''(1) = \langle 2, 2, 0\rangle.$$

So

$$\vec{r}\,'(1) \times \vec{r}\,''(1) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 2 & 0 \end{vmatrix} = \langle -4, 4, -2\rangle \quad \text{and} \quad \|\vec{r}\,'(1) \times \vec{r}\,''(1)\| = \sqrt{16+16+4} = 6.$$

The curvature is

$$\kappa = \frac{\|\vec{r}\,'(1) \times \vec{r}\,''(1)\|}{\|\vec{r}\,'(1)\|^3} = \frac{6}{3^3} = \frac{2}{9}.$$

The point on the curve is $\vec{r}(1) = \left(\frac{4}{3}, 2, 7\right)$. Therefore the equation of the osculating circle is

$$(x, y, z) = \left\langle \frac{4}{3}, 2, 7 \right\rangle + \frac{9}{2} \cdot \frac{1}{3} \langle 2, 1, -2 \rangle + \frac{9}{2} \cdot \frac{1}{3} \langle 1, 2, 2 \rangle \cos t + \frac{9}{2} \cdot \frac{1}{3} \langle 2, 1, -2 \rangle \sin t =$$

$$\left\langle \frac{13}{3} + \frac{3}{2} \cos t + 3 \sin t, \frac{7}{2} + 3 \cos t + \frac{3}{2} \sin t, 4 + 3 \cos t - 3 \sin t \right\rangle. \quad \square$$

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3. Find the domain and range of $f(x, y, z) = \dfrac{z^2 + 1}{\sqrt{1 - x^2 - y^2}}$.

The function is defined for $1 - x^2 - y^2 > 0$. Therefore, the domain is the set of points (x, y, z) such that $x^2 + y^2 < 1$ — that is, the interior of the cylinder $x^2 + y^2 = 1$ of radius 1 whose axis is the z-axis.

To find the range, note that $z^2 + 1 \geq 1$. Also,

$$1 - x^2 - y^2 \leq 1, \quad \text{and} \quad \sqrt{1 - x^2 - y^2} \leq 1, \quad \text{so} \quad \frac{1}{\sqrt{1 - x^2 - y^2}} \geq 1.$$

Hence,

$$f(x, y, z) = \frac{z^2 + 1}{\sqrt{1 - x^2 - y^2}} \geq 1 \cdot 1 = 1.$$

This shows that every output of f is greater than or equal to 1.

On the other hand, suppose $k \geq 1$. Then

$$f(0, 0, \sqrt{k - 1}) = \frac{(\sqrt{k - 1})^2 + 1}{\sqrt{1 - 0 - 0}} = k.$$

This shows that every number greater than or equal to 1 is an output of f.

Hence, the range of f is the set of numbers w such that $w \geq 1$. $\square$

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4. (a) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^4 + 5y^4}{x^4 + 3x^2y^2 + y^4}$ is undefined.

If you approach $(0, 0)$ along the x-axis ($y = 0$), you get

$$\lim_{(x,y)\to(0,0)} \frac{3x^4 + 5y^4}{x^4 + 3x^2y^2 + y^4} = \lim_{(x,y)\to(0,0)} \frac{3x^4}{x^4} = \lim_{(x,y)\to(0,0)} 3 = 3.$$

If you approach $(0, 0)$ along the line $y = x$, you get

$$\lim_{(x,y)\to(0,0)} \frac{3x^4 + 5y^4}{x^4 + 3x^2y^2 + y^4} = \lim_{(x,y)\to(0,0)} \frac{3x^4 + 5x^4}{x^4 + 3x^4 + x^4} = \lim_{(x,y)\to(0,0)} \frac{8x^4}{5x^4} = \lim_{(x,y)\to(0,0)} \frac{8}{5} = \frac{8}{5}.$$

Since the function approaches different values as you approach $(0, 0)$ in different ways, the limit is undefined. $\square$

(b) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{x^4 + 3x^2y^2 + y^4}$ is defined and find its value.

$$\left| \frac{x^4y^4}{x^4 + 3x^2y^2 + y^4} \right| \leq \left| \frac{x^4y^4}{x^4} \right| = |y^4| \to 0 \quad \text{as} \quad (x, y) \to (0, 0).$$

Therefore,

$$\lim_{(x,y)\to(0,0)} \left|\frac{x^4y^4}{x^4+3x^2y^2+y^4}\right| = 0.$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{x^4+3x^2y^2+y^4} = 0. \quad \square$$

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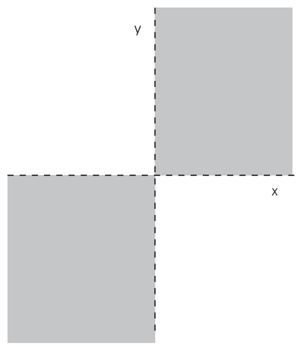
5. For what points (x, y) is the function $f(x, y) = \ln(xy)$ continuous?

The function is continuous wherever it's defined. For $\ln(xy)$ to be defined, I must have $xy > 0$. Therefore, either x and y are both positive or x and y are both negative.

Hence, f is continuous for (x, y) in the first quadrant or the third quadrant of the x-y-plane.



$\square$

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6. Find the tangent plane to the surface

$$x = u^2 - 3v^2, \quad y = \frac{4u}{v}, \quad z = 2u^2v^3$$

for $u = 1$ and $v = 1$.

When $u = 1$ and $v = 1$, $x = -2$, $y = 4$, and $z = 2$. The point of tangency is $(-2, 4, 2)$.
Next,

$$\vec{T}_u = \left\langle 2u, \frac{4}{v}, 4uv^3 \right\rangle \quad \text{and} \quad \vec{T}_v = \left\langle -6v, -\frac{4u}{v^2}, 6u^2v^2 \right\rangle.$$

Thus,

$$\vec{T}_u(1,1) = \langle 2, 4, 4\rangle \quad \text{and} \quad \vec{T}_v(1,1) = \langle -6, -4, 6\rangle.$$

The normal vector is given by

$$\vec{T}_u(1,1) \times \vec{T}_v(1,1) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 4 & 4 \\ -6 & -4 & 6 \end{vmatrix} = \langle 40, -36, 16\rangle.$$

The tangent plane is

$$40(x+2) - 36(y-4) + 16(z-2) = 0, \quad \text{or} \quad 10x - 9y + 4z = -48. \quad \square$$

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7. Use a linear approximation to $z = f(x, y) = x^2 - y^2$ at the point $(2, 1)$ to approximate $f(1.9, 1.1)$.

$f(2, 1) = 3$, so the point of tangency is $(2, 1, 3)$. A normal vector for a function $z = f(x, y)$ is given by

$$\vec{N} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right\rangle = \langle 2x, -2y, -1 \rangle, \quad \vec{N}(2, 1) = \langle 4, -2, -1 \rangle.$$

Hence, the tangent plane is

$$4(x - 2) - 2(y - 1) - (z - 3) = 0, \quad \text{or} \quad z = 3 + 4(x - 2) - 2(y - 1).$$

Substitute $x = 1.9$ and $y = 1.1$:

$$z = 3 + 4(-0.1) - 2(0.1) = 2.4. \quad \Box$$

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8. Find the gradient of $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2 + 1}}$ and show that it always points toward the origin.

$$\nabla f = \left\langle \frac{-x}{(x^2 + y^2 + z^2 + 1)^{3/2}}, \frac{-y}{(x^2 + y^2 + z^2 + 1)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2 + 1)^{3/2}} \right\rangle =$$

$$\frac{-1}{(x^2 + y^2 + z^2 + 1)^{3/2}} \langle x, y, z \rangle.$$

$\langle x, y, z \rangle$ is the **radial vector** from the origin $(0, 0, 0)$ to the point (x, y, z). Since ∇f is a negative multiple of this vector ∇f always points *inward* toward the origin. $\Box$

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9. Find the rate of change of $f(x, y, z) = xy - yz + xz$ at the point $(1, -2, -2)$ in the direction toward the origin. Is f increasing or decreasing in this direction?

First, compute the gradient at the point:

$$\nabla f = \langle y + z, x - z, -y + x \rangle, \quad \nabla f(1, -2, -2) = \langle -4, 3, 3 \rangle.$$

Next, determine the direction vector. The point is $P(1, -2, -2)$, so the direction toward the origin $Q(0, 0, 0)$ is

$$\overrightarrow{PQ} = \langle -1, 2, 2 \rangle.$$

Make this into a unit vector by dividing by its length:

$$\frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{1}{3} \langle -1, 2, 2 \rangle.$$

Finally, take the dot product of the unit vector with the gradient:

$$Df_{\vec{v}}(1, -2, -2) = \nabla f(1, -2, -2) \cdot \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} \langle -4, 3, 3 \rangle \cdot \frac{1}{3} \langle -1, 2, 2 \rangle = \frac{16}{3}.$$

f is increasing in this direction, since the directional derivative is positive. $\Box$

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10. Calvin Butterball sits in his go-cart on the surface

$$z = x^3 - 2x^2y + x^2 + xy^2 - 2y^3 + y^2$$

at the point $(1, 1, 0)$. If his go-cart is pointed in the direction of the vector $\vec{v} = \langle 15, -8\rangle$, at what rate will it roll downhill?

The rate at which he rolls is given by the directional derivative. The gradient is

$$\nabla f = \langle 3x^2 - 6xy + 2x + y^2, -2x^2 + 2xy - 6y^2 + 2y\rangle, \quad \text{and} \quad \nabla f(1, 1) = \langle 0, -4\rangle.$$

Since $\|\langle 15, -8\rangle\| = 17$,

$$Df_{\vec{v}}(1, 1) = \langle 0, -4\rangle \cdot \frac{\langle 15, -8\rangle}{17} = \frac{32}{17} \approx 1.88235. \quad \square$$

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11. Find the tangent plane to $x^2 - y^2 + 2yz + z^5 = 6$ at the point $(2, 1, 1)$.

Write $w = x^2 - y^2 + 2yz + z^5 - 6$. (Take the original surface and drag everything to one side of the equation.) The original surface is $w = 0$, so it's a level surface of w. Since the gradient ∇w is perpendicular to the level surfaces of w, ∇w must be perpendicular to the original surface.

The gradient is

$$\nabla w = \langle 2x, -2y + 2z, 2y + 5z^4\rangle, \quad \nabla w(2, 1, 1) = \langle 4, 0, 7\rangle.$$

The vector $\langle 4, 0, 7\rangle$ is perpendicular to the tangent plane. Hence, the plane is

$$4(x - 2) + 0 \cdot (y - 1) + 7(z - 1) = 0, \quad \text{or} \quad 4x + 7z = 15. \quad \square$$

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12. The rate of change of $f(x, y)$ at $(1, -1)$ is 2 in the direction *toward* $(5, -1)$ and is $\dfrac{6}{5}$ in the direction of the vector $\langle -3, -4\rangle$. Find $\nabla f(1, -1)$.

The direction from $(1, -1)$ *toward* the point $(5, -1)$ is given by the vector $\langle 4, 0\rangle$. This vector has length 4, so

$$2 = \nabla f(1, -1) \cdot \frac{\langle 4, 0\rangle}{4} = \langle f_x, f_y\rangle \cdot \frac{\langle 4, 0\rangle}{4} = f_x.$$

The vector $\langle -3, -4\rangle$ has length 5, so

$$\frac{6}{5} = \nabla f(1, -1) \cdot \frac{\langle -3, -4\rangle}{5} = \langle f_x, f_y\rangle \cdot \frac{\langle -3, -4\rangle}{5} = -\frac{3}{5} f_x - \frac{4}{5} f_y.$$

Thus, $6 = -3f_x - 4f_y$.

I have two equations involving f_x and f_y. Solving simultaneously, I obtain $f_x = 2$ and $f_y = -3$. Hence, $\nabla f(1, -1) = \langle 2, -3\rangle$. $\square$

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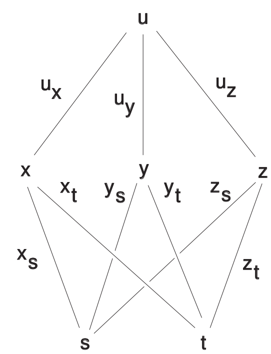
13. Let r and θ be the standard polar coordinates variables. Use the Chain Rule to find $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$, for $f(x, y) = xe^x + e^y$.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = (xe^x + e^x)(\cos\theta) + (e^y)(\sin\theta),$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = (xe^x + e^x)(-r\sin\theta) + (e^y)(r\cos\theta). \quad \square$$

14. Suppose $u = f(x, y, z)$ and $x = \phi(s, t)$, $y = \psi(s, t)$, $z = \mu(s, t)$. Use the Chain Rule to write down an expression for $\frac{\partial u}{\partial t}$.

This diagram shows the dependence of the variables.



There are 3 paths from u to t, which give rise to the 3 terms in the following sum:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}. \quad \square$$

15. Suppose that $w = f(x, y)$, $x = g(r, s, t)$, and $y = h(r, t, s)$. Use the Chain Rule to find an expression for $\frac{\partial^2 f}{\partial t^2}$. By the Chain Rule,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

Next, differentiate with respect to t, applying the Product Rule to the terms on the right:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right).$$

Since $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are functions of x and y, I must apply the Chain Rule in computing their derivatives with respect to t. I get

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial t}\left(\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)\frac{\partial y}{\partial t}\right) + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t}\left(\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\frac{\partial y}{\partial t}\right) =$$

$$\frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial y}{\partial t}\right) + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t}\left(\frac{\partial^2 w}{\partial x \partial y}\frac{\partial x}{\partial t} + \frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial t}\right). \quad \square$$

16. Locate and classify the critical points of

$$z = x^2 y - 4xy + \frac{1}{3}y^3 - \frac{3}{2}y^2.$$

$$\frac{\partial z}{\partial x} = 2xy - 4y, \quad \frac{\partial z}{\partial y} = x^2 - 4x + y^2 - 3y,$$

$$\frac{\partial^2 z}{\partial x^2} = 2y, \quad \frac{\partial^2 z}{\partial x \partial y} = 2x - 4, \quad \frac{\partial^2 z}{\partial y^2} = 2y - 3.$$

Set the first partials equal to 0:

$$(1) \qquad 2xy - 4y = 0, \quad (x-2)y = 0,$$

$$(2) \qquad x^2 - 4x + y^2 - 3y = 0.$$

Solve simultaneously:

$$(1) \quad (x-2)y = 0$$

$x = 2$:

$$(2) \quad x^2 - 4x + y^2 - 3y = 0$$
$$y^2 - 3y - 4 = 0$$
$$(y-4)(y+1) = 0$$

$y = 4$: $(2, 4)$; $y = -1$: $(2, -1)$

$y = 0$:

$$(2) \quad x^2 - 4x + y^2 - 3y = 0$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0$$

$x = 0$: $(0, 0)$; $x = 4$: $(4, 0)$

Test the critical points:

point	z_{xx}	z_{yy}	z_{xy}	Δ	result
$(2, 4)$	8	5	0	40	min
$(2, -1)$	-2	-5	0	10	max
$(0, 0)$	0	-3	-4	-16	saddle
$(4, 0)$	0	-3	4	-16	saddle

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17. Find the points on the sphere $x^2 + y^2 + z^2 = 36$ which are closest to and farthest from the point $(4, -3, 12)$.

The (square of the) distance from (x, y, z) to $(4, -3, 12)$ is

$$w = (x-4)^2 + (y+3)^2 + (z-12)^2.$$

The constraint is $g(x, y, z) = x^2 + y^2 + z^2 - 36 = 0$.
The equations to be solved are

$$(1) \qquad 2(x-4) = 2x\lambda, \quad x - 4 = x\lambda,$$

$$(2) \qquad 2(y+3) = 2y\lambda, \quad y + 3 = y\lambda,$$

$$(3) \qquad 2(z-12) = 2z\lambda, \quad z - 12 = z\lambda.$$

Note that if $x = 0$ in the first equation, the equation becomes $-4 = 0$, which is impossible. Therefore, $x \neq 0$, and *I may divide by* x.

Solve simultaneously:

$$\begin{aligned}
&(1) & x - 4 &= x\lambda \\
& & \lambda &= \frac{x-4}{x} \\
&(2) & y + 3 &= y\lambda \\
& & y + 3 &= \frac{y(x-4)}{x} \\
& & xy + 3x &= yx - 4y \\
& & y &= -\frac{3}{4}x \\
&(3) & z - 12 &= z\lambda \\
& & z - 12 &= \frac{z(x-4)}{x} \\
& & xz - 12x &= xz - 4z \\
& & z &= 3x \\
&(4) & x^2 + y^2 + z^2 &= 36 \\
& & x^2 + \frac{9}{16}x^2 + 9x^2 &= 36 \\
& & 169x^2 &= 576 \\
& & x^2 &= \frac{576}{169}
\end{aligned}$$

$\swarrow$ $\searrow$

$$x = \frac{24}{13} \quad y = -\frac{18}{13} \quad z = \frac{72}{13} \quad \left(\frac{24}{13}, -\frac{18}{13}, \frac{72}{13}\right)$$

$$x = -\frac{24}{13} \quad y = \frac{18}{13} \quad z = -\frac{72}{13} \quad \left(-\frac{24}{13}, \frac{18}{13}, -\frac{72}{13}\right)$$

Test the points:

	$\left(\frac{24}{13}, -\frac{18}{13}, \frac{72}{13}\right)$	$\left(-\frac{24}{13}, \frac{18}{13}, -\frac{72}{13}\right)$
$w(x,y,z)$	49	361

$\left(\frac{24}{13}, -\frac{18}{13}, \frac{72}{13}\right)$ is closest to $(4, -3, 12)$ and $\left(-\frac{24}{13}, \frac{18}{13}, -\frac{72}{13}\right)$ is farthest from $(4, -3, 12)$. □

To be conscious that you are ignorant is a great step to knowledge. - Benjamin Disraeli