Math 261-00
4–22–2004

Review Sheet for the Final

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. Find the parametric and symmetric equations of the line which passes through the points $P(3,4,6)$ and $Q(-1,3,2)$.

2. Determine whether the following lines are parallel, skew, or intersect. If they intersect, find the point of intersection.

$$x = 2+t, \quad y = 3-t, \quad z = 4+2t,$$

$$x = 1+s, \quad y = 6-2s, \quad z = 3s.$$

3. Show that the following lines are skew, and find the distance between them.

$$x = 2-t, \quad y = 3+4t, \quad z = 2t,$$

$$x = -1+u, \quad y = 2, \quad z = -1+2u.$$

problem Show that the following lines are parallel, and find the distance between them.

$$x = t, \quad y = 1+t, \quad z = 1-t,$$

$$x = 1-2s, \quad y = 1-2s, \quad z = 2s.$$

4. Find the point of intersection of the line

$$x = 3+t, \quad y = 5+2t, \quad z = 2-2t \quad \text{and the plane} \quad 2x+y-z = 3.$$

5. Find the equation of the plane containing the points $P(4,-3,1)$, $Q(6,-4,7)$, and $R(1,2,2)$.

6. Let $f(x,y,z) = x^2y^2 - 2xyz + z^2$.

(a) Find the rate of most rapid increase at $(1,-1,1)$.

(b) Find a unit vector pointing in the direction of most rapid increase.

7. Find the rate of change of $f(x,y) = \dfrac{x^2}{y^2} + 5x^2 - xy$ at the point $(1,1)$ in the direction:

(a) Given by the vector $\vec{v} = \langle 3,-4\rangle$.

(b) Toward the point $(9,-14)$.

8. Suppose $w = f(x,y,z)$, $x = p(u,v)$, $y = q(u,v)$, and $z = r(u,v)$.

(a) Use the Chain Rule to find an expression for $\dfrac{\partial w}{\partial u}$.

(b) Use the Chain Rule to find an expression for $\dfrac{\partial^2 w}{\partial u^2}$.

9. Locate and classify the critical points of

$$z = 2x^3 - 3x^2y + \frac{4}{3}y^3 - 4y + 6.$$

10. Find the dimensions of the rectangular box with no top having maximal volume and surface area 48.

11. Find the unit tangent, the unit normal, the curvature, and the equation of the osculating circle for the curve

$$\vec{\sigma}(t) = \langle (t+1)^2, t^3 + 2t + 1 \rangle, \quad \text{at the point} \quad t = 1.$$

12. Find the volume of the region in the first octant cut off by the plane $2x + y + 2z = 8$.

13. Compute $\displaystyle\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} e^{3x - x^3}\, dx\, dy$.

14. Compute $\displaystyle\iiint_R (6x + 4y)\, dV$, where R is the region in the first octant bounded above by $z = y^2$ and bounded on the side by $x + y = 1$.

15. The solid bounded above by $z = 2 - 2x^2 - 2y^2$ and below by $z = x^2 + y^2 - 1$ has density $\rho = 2$. Find the mass and the center of mass.

16. (a) Parametrize the surface generated by revolving $y = x^2$, $0 \le x \le 1$, about the x-axis.

(b) Find the area of the surface.

17. A wire is made of the three segments connecting the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. The density of the wire is $\delta = x + y + z$. Find its mass.

18. Let

$$\vec{\sigma}(t) = \left\langle te^{t-1}, t^3, \sin\left(\frac{\pi t}{2}\right) \right\rangle, \quad 0 \le t \le 1.$$

Compute

$$\int_{\vec{\sigma}} (y - z^2)\, dx + (x - 2y + 2yz)\, dy + (y^2 + 2z - 2xz)\, dz.$$

19. Let

$$\vec{F} = \left\langle \frac{x}{x^2 + y^2 + z^2 + 1}, \frac{y}{x^2 + y^2 + z^2 + 1}, \frac{z}{x^2 + y^2 + z^2 + 1} \right\rangle,$$

and let $\vec{\sigma}(t)$ be *any* path from *any* point on the sphere $x^2 + y^2 + z^2 = 1$ to *any* point on the sphere $x^2 + y^2 + z^2 = 5$. Compute $\displaystyle\int_{\vec{\sigma}} \vec{F} \cdot d\vec{s}$.

20. Use Green's Theorem to show that the area of the ellipse

$$x = a\cos t, \quad y = b\sin t, \qquad 0 \le t \le 2\pi$$

is πab.

21. Let $\vec{\sigma}$ be the path which starts at $(2, 0)$, goes around the circle $x^2 + y^2 = 4$ in the counterclockwise direction, traverses the segment from $(2, 0)$ to $(1, 0)$, goes around the circle $x^2 + y^2 = 1$ in the clockwise direction, and traverses the segment from $(1, 0)$ to $(2, 0)$. Compute $\displaystyle\int_{\vec{\sigma}} -y\, dx + x\, dy$.

22. Compute the circulation of $\vec{F} = \langle yz, xz, -xy \rangle$ counterclockwise (as viewed from above) around the triangle with vertices $A(1, 2, 1)$, $B(2, 1, 4)$, and $C(-3, 1, 1)$.

23. Let $\vec{\sigma}$ be the curve of intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$, traversed counterclockwise as viewed from above. Compute the circulation of $\vec{F} = \langle x^2 y^3, 1, z \rangle$ around $\vec{\sigma}$:

(a) Directly, by parametrizing the curve and computing the line integral.

(b) Using Stokes' theorem.

24. Let R be the solid region in the first octant cut off by the sphere $x^2 + y^2 + z^2 = 1$. Compute the flux out through the boundary of R of the vector field

$$\vec{F} = \left\langle x^3 + yz, y^3 - xz, z^3 + 2xy \right\rangle.$$

Solutions to the Review Sheet for the Final

1. Find the parametric and symmetric equations of the line which passes through the points $P(3,4,6)$ and $Q(-1,3,2)$.

I need a point on the line and a vector parallel to the line.
For the point on the line, I can take either P or Q; I'll use $P(3,4,6)$.
Since P and Q are on the line, the vector $\vec{PQ} = \langle -4, -1, -4 \rangle$ is parallel to the line.
Hence, the parametric equations for the line are

$$x - 3 = -4t, \quad y - 4 = -t, \quad z - 6 = -4t.$$

The symmetric equations are

$$\frac{x-3}{-4} = -(y-4) = \frac{z-6}{-4}. \quad \square$$

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2. Determine whether the following lines are parallel, skew, or intersect. If they intersect, find the point of intersection.

$$x = 2 + t, \quad y = 3 - t, \quad z = 4 + 2t,$$

$$x = 1 + s, \quad y = 6 - 2s, \quad z = 3s.$$

The vector $\langle 1, -1, 2 \rangle$ is parallel to the first line. The vector $\langle 1, -2, 3 \rangle$ is parallel to the second line. The vectors aren't multiples of one another, so the vectors aren't parallel. Therefore, the lines aren't parallel.

Next, I'll check whether the lines intersect.
Solve the x-equations simultaneously:

$$2 + t = 1 + s, \quad s = 1 + t.$$

Set the y-expressions equal, then plug in $s = 1 + t$ and solve for t:

$$3 - t = 6 - 2s \quad 3 - t = 6 - 2(1 + t), \quad 3 - t = 4 - 2t, \quad t = 1.$$

Therefore, $x = 1 + t = 2$.
Check the values for consistency by plugging them into the z-equations:

$$z = 4 + 2t = 6, \quad z = 3s = 6.$$

The equations are consistent, so the lines intersect. If I plug $t = 1$ into the x-y-z equations, I obtain $x = 3$, $y = 2$, and $z = 6$. The lines intersect at $(3, 2, 6)$. $\square$

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3. Show that the following lines are skew, and find the distance between them.

$$x = 2 - t, \quad y = 3 + 4t, \quad z = 2t,$$

$$x = -1 + u, \quad y = 2, \quad z = -1 + 2u.$$

The vector $\langle -1, 4, 2\rangle$ is parallel to the first line, and the vector $\langle 1, 0, 2\rangle$ is parallel to the second. The vectors are not multiples of each other, so the vectors aren't parallel. Hence, the lines aren't parallel.

If the lines intersect, the distance between them is 0. Hence, I'll just go on to find the distance between the lines. If the distance is nonzero, the lines can't intersect, so they must be skew.

You can think of skew lines as lying in parallel planes. The idea is to find a vector perpendicular to the two lines (or the two planes). Next, find a point P on the first line and a point Q on the second. Finally, the distance will be the absolute value of $\operatorname{comp}_{\vec{v}} \overrightarrow{PQ}$.

I can get a vector $\vec{v}$ perpendicular to both lines by taking the cross product of the vectors parallel to the two lines:

$$\langle -1, 4, 2\rangle \times \langle 1, 0, 2\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 4 & 2 \\ 1 & 0 & 2 \end{vmatrix} = \langle 8, 4, -4\rangle.$$

Set $t = 0$ in the first line to obtain $P(2, 3, 0)$; set $u = 0$ in the second line to obtain $Q(-1, 2, -1)$. Then $\overrightarrow{PQ} = \langle -3, -1, -1\rangle$. Hence,

$$\operatorname{comp}_{\vec{v}} \overrightarrow{PQ} = \frac{\langle -3, -1, -1\rangle \cdot \langle 8, 4, -4\rangle}{\|\langle 8, 4, -4\rangle\|} = -\sqrt{6}.$$

The distance is $\sqrt{6} \approx 2.44949$. $\square$

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problem Show that the following lines are parallel, and find the distance between them.

$$x = t, \quad y = 1 + t, \quad z = 1 - t,$$

$$x = 1 - 2s, \quad y = 1 - 2s, \quad z = 2s.$$

The vector $\langle 1, 1, -1\rangle$ is parallel to the first line; the vector $\langle -2, -2, 2\rangle$ is parallel to the second line. The second vector is -2 times the first, so the vectors are parallel. Hence, the lines are parallel.

Next, I'll find the distance between the lines. If I set $t = 0$, I find that $P(0, 1, 1)$ lies on the first line; likewise, setting $s = 0$, I find that $Q(1, 1, 0)$ lies on the second line. Now $\overrightarrow{PQ} = \langle 1, 0, -1\rangle$; projecting this onto the first line's vector $\vec{v} = \langle 1, 1, -1\rangle$, I obtain

$$\operatorname{comp}_{\vec{v}} \overrightarrow{PQ} = \frac{\langle 1, 0, -1\rangle \cdot \langle 1, 1, -1\rangle}{\|\langle 1, 1, -1\rangle\|} = \frac{2}{\sqrt{3}}.$$

I find the distance between the lines using Pythagoras' theorem:

$$\text{distance} = \left[\|\vec{PQ}\|^2 - \left(\operatorname{comp}_{\vec{v}} \overrightarrow{PQ}\right)^2\right]^{1/2} = \sqrt{\frac{2}{3}} \approx 0.81650. \quad \square$$

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4. Find the point of intersection of the line

$$x = 3 + t, \quad y = 5 + 2t, \quad z = 2 - 2t \quad \text{and the plane} \quad 2x + y - z = 3.$$

Plug the expressions for x, y, and z from the line into the equation of the plane:

$$2(3 + t) + (5 + 2t) - (2 - 2t) = 3, \quad 6t = -6, \quad t = -1.$$

Plugging $t = -1$ into the equations for x, y, and z gives $x = 2$, $y = 3$, and $z = 4$. The point of intersection is $(2, 3, 4)$. $\square$

5. Find the equation of the plane containing the points $P(4,-3,1)$, $Q(6,-4,7)$, and $R(1,2,2)$.

The vectors $\overrightarrow{PQ} = \langle 2,-1,6\rangle$ and $\overrightarrow{PR} = \langle -3,5,1\rangle$ lie in the plane, so their cross product is perpendicular to the plane. The cross product is

$$\langle 2,-1,6\rangle \times \langle -3,5,1\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 6 \\ -3 & 5 & 1 \end{vmatrix} = \langle -31,-20,7\rangle.$$

Since the point $P(4,-3,1)$ lies on the plane, the plane is

$$-31(x-4) - 20(y+3) + 7(z-1) = 0, \quad \text{or} \quad -31x - 20y + 7z = -57. \quad \square$$

6. Let $f(x,y,z) = x^2y^2 - 2xyz + z^2$.

(a) Find the rate of most rapid increase at $(1,-1,1)$.

$$\nabla f = \langle 2xy^2 - 2yz, 2x^2y - 2xz, -2xy + 2z\rangle, \quad \nabla f(1,-1,1) = \langle 4,-4,4\rangle.$$

The rate of most rapid increase is

$$\|\nabla f\| = \sqrt{4^2 + (-4)^2 + 4^2} = \sqrt{48} = 4\sqrt{3}. \quad \square$$

(b) Find a unit vector pointing in the direction of most rapid increase.

The gradient points in the direction of most rapid increase. Therefore, a unit vector pointing in the direction of most rapid increase is given by

$$\frac{\nabla f}{\|\nabla f\|} = \frac{\langle 4,-4,4\rangle}{4\sqrt{3}} = \frac{1}{\sqrt{3}}\langle 1,-1,1\rangle. \quad \square$$

7. Find the rate of change of $f(x,y) = \dfrac{x^2}{y^2} + 5x^2 - xy$ at the point $(1,1)$ in the direction:

(a) Given by the vector $\vec{v} = \langle 3,-4\rangle$.

$$\nabla f = \left\langle \frac{2x}{y^2} + 10x - y, -\frac{2x^2}{y^3} - x \right\rangle, \quad \nabla f(1,1) = \langle 11,-3\rangle.$$

Therefore,

$$Df_{\vec{v}}(1,1) = \nabla f(1,1) \cdot \frac{\langle 3,-4\rangle}{\|\langle 3,-4\rangle\|} = \langle 11,-3\rangle \cdot \frac{\langle 3,-4\rangle}{5} = 9. \quad \square$$

(b) Toward the point $(9,-14)$.

The vector from $(1,1)$ to $(9,-14)$ is $\vec{w} = \langle 8,-15\rangle$.
So

$$Df_{\vec{w}}(1,1) = \nabla f(1,1) \cdot \frac{\langle 8,-15\rangle}{\|\langle 8,-15\rangle\|} = \langle 11,-3\rangle \cdot \frac{\langle 8,-15\rangle}{17} = \frac{133}{17}. \quad \square$$

8. Suppose $w = f(x, y, z)$, $x = p(u, v)$, $y = q(u, v)$, and $z = r(u, v)$.

(a) Use the Chain Rule to find an expression for $\dfrac{\partial w}{\partial u}$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}. \quad \square$$

(b) Use the Chain Rule to find an expression for $\dfrac{\partial^2 w}{\partial u^2}$.

$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}\right) =$$

$$\frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial x}{\partial u}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y \partial x}\frac{\partial y}{\partial u} + \frac{\partial^2 w}{\partial z \partial x}\frac{\partial z}{\partial u}\right) + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial u^2} + \frac{\partial y}{\partial u}\left(\frac{\partial^2 w}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial u} + \frac{\partial^2 w}{\partial z \partial y}\frac{\partial z}{\partial u}\right) +$$

$$\frac{\partial w}{\partial z}\frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial u}\left(\frac{\partial^2 w}{\partial x \partial z}\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y \partial z}\frac{\partial y}{\partial u}\frac{\partial^2 w}{\partial z^2}\frac{\partial z}{\partial u}\right). \quad \square$$

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9. Locate and classify the critical points of

$$z = 2x^3 - 3x^2 y + \frac{4}{3}y^3 - 4y + 6.$$

$$z_x = 6x^2 - 6xy, \quad z_y = -3x^2 + 4y^2 - 4,$$
$$z_{xx} = 12x - 6y, \quad z_{yy} = 8y, \quad z_{xy} = -6x.$$

Set the first-order partials equal to 0:

$$(1) \qquad 6x^2 - 6xy = 0, \quad x(x - y) = 0,$$

$$(2) \qquad -3x^2 + 4y^2 - 4 = 0.$$

Solve simultaneously:

$$(1) \qquad x(x - y) = 0$$

$\swarrow$ $\searrow$

$$\begin{array}{rcl} & x = 0 & \\ (2) & -3x^2 + 4y^2 - 4 = 0 & \\ & 4y^2 - 4 = 0 & \\ & y^2 = 1 & \end{array} \qquad \begin{array}{c} x = y \\ -3x^2 + 4y^2 - 4 = 0 \\ y^2 - 4 = 0 \\ y^2 = 4 \end{array}$$

$\swarrow$ $\searrow$ $\swarrow$ $\searrow$

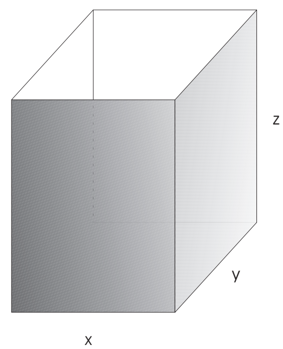
$$\begin{array}{c} y = 1 \\ (0, 1) \end{array} \qquad \begin{array}{c} y = -1 \\ (0, -1) \end{array} \qquad \begin{array}{c} y = 2 \\ x = 2 \\ (2, 2) \end{array} \qquad \begin{array}{c} y = -2 \\ x = -2 \\ (-2, -2) \end{array}$$

Test the critical points:

point	z_{xx}	z_{yy}	z_{xy}	Δ	result
$(0, 1)$	-6	8	0	-48	saddle
$(0, -1)$	6	-8	0	-48	saddle
$(2, 2)$	12	16	-12	48	min
$(-2, -2)$	-12	-16	12	48	max

$\square$

10. Find the dimensions of the rectangular box with no top having maximal volume and surface area 48.



Let x and y be the dimensions of the base, and let z be the height. I want to find the maximum value of $f(x, y) = xyz$ subject to the constraint $48 = xy + 2xz + 2yz$. Write $g(x, y, z) = xy + 2xz + 2yz - 48 = 0$.

I obtain the equations

$$yz = \lambda(y + 2z), \tag{1}$$

$$xz = \lambda(x + 2z), \tag{2}$$

$$xy = \lambda(2x + 2y), \tag{3}$$

$$48 = xy + 2xz + 2yz. \tag{4}$$

Before continuing, note that since x, y, and z are the dimensions of a box, they can't be 0 or negative.

In addition, I may assume that $x, y, z > 0$. For $x = 2$, $y = 2$, and $z = \dfrac{11}{2}$ satisfies the constraint and gives a box of volume 6. So I can certainly do better (in terms of getting a larger volume) than to have one of the dimensions equal 0, which would give a box a volume 0. This implies that I may divide by x, y, or z, and I'll do so below without further comment.

Since $x, y, z \neq 0$, I may assume that $y + 2z \neq 0$. For if $y + 2z = 0$, then the first equation gives $0 = yz$, which would imply that $y = 0$ or $z = 0$. Likewise, I may assume that $x + 2z \neq 0$, and $2x + 2y \neq 0$. This implies that I may divide by $y + 2z$, and I'll do so below without further comment.

Now I'll solve the equations simultaneously.

$$\begin{array}{rr@{}l}
(1) & yz &= \lambda(y + 2z) \\
& \lambda &= \dfrac{yz}{y + 2z} \\
(2) & xz &= \lambda(x + 2z) \\
& xz &= \dfrac{yz(x + 2z)}{y + 2z} \\
& xz(y + 2z) &= yz(x + 2z) \\
& xyz + 2xz^2 &= xyz + 2yz^2 \\
& 2xz^2 &= 2yz^2 \\
& x &= y \\
(3) & xy &= \lambda(2x + 2y) \\
& xy &= \dfrac{yz(2x + 2y)}{y + 2z} \\
& xy(y + 2z) &= yz(2x + 2y) \\
& xy^2 + 2xyz &= 2xyz + 2y^2 z \\
& x &= 2z \\
& z &= \dfrac{x}{2} \\
(4) & 48 &= xy + 2xz + 2yz \\
& 48 &= x^2 + x^2 + x^2 \\
& 3x^2 &= 48 \\
& x &= 4 \\
& y &= 4 \\
& z &= 2
\end{array}$$

The dimensions $x = 4$, $y = 4$, and $z = 2$ maximize the volume. (I can satisfy the constraint and make the volume arbitrarily small by making one of the dimensions sufficiently small. Thus, the point $(4, 4, 2)$ can't give a min.) $\square$

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11. Find the unit tangent, the unit normal, the curvature, and the equation of the osculating circle for the curve

$$\vec{\sigma}(t) = \langle (t + 1)^2, t^3 + 2t + 1 \rangle, \quad \text{at the point} \quad t = 1.$$

$$\vec{\sigma}\,'(t) = \langle 2(t + 1), 3t^2 + 2 \rangle, \quad \vec{\sigma}\,'(1) = \langle 4, 5 \rangle.$$

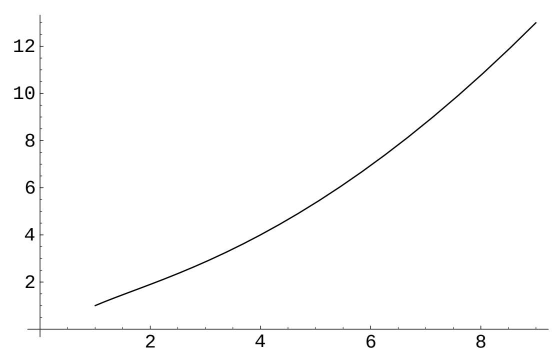
The unit tangent is

$$\vec{T}(1) = \frac{\vec{\sigma}\,'(1)}{\|\vec{\sigma}\,'(1)\|} = \frac{1}{\sqrt{41}} \langle 4, 5 \rangle.$$

The unit vectors

$$\frac{1}{\sqrt{41}} \langle 5, -4 \rangle \quad \text{and} \quad \frac{1}{\sqrt{41}} \langle -5, 4 \rangle$$

are clearly perpendicular to $\vec{T}(1)$. Here's a picture of the curve for $0 \le t \le 2$:



The unit normal must point upward, so its y-component must be positive. Therefore,

$$\vec{N}(1) = \frac{1}{\sqrt{41}}\langle -5, 4\rangle.$$

For the curvature, I'll use the formula

$$\kappa = \frac{|x'(1)y''(1) - x''(1)y'(1)|}{(x'(1)^2 + y'(1)^2)^{3/2}}.$$

In this case,

$$x' = 2(t+1), \quad x'(1) = 4, \quad y' = 3t^2 + 2, \quad y'(1) = 5,$$
$$x'' = 2, \quad x''(1) = 2, \quad y'' = 6t, \quad y''(1) = 6.$$

Therefore,

$$\kappa = \frac{|(4)(6) - (2)(5)|}{(4^2 + 5^2)^{3/2}} = \frac{14}{41\sqrt{41}}.$$

The point on the curve is $\vec{\sigma}(1) = \langle 4, 4\rangle$ and the radius of curvature is $R = \dfrac{41\sqrt{41}}{14}$. The osculating circle is

$$\langle x, y\rangle = \langle 4, 4\rangle + \frac{41\sqrt{41}}{14}\cdot\frac{1}{\sqrt{41}}\langle -5, 4\rangle + \frac{41\sqrt{41}}{14}\cdot\frac{1}{\sqrt{41}}\langle 4, 5\rangle\cos t + \frac{41\sqrt{41}}{14}\cdot\frac{1}{\sqrt{41}}\langle -5, 4\rangle\sin t =$$
$$\langle 4, 4\rangle + \frac{41}{14}\langle -5, 4\rangle + \frac{41}{14}\langle 4, 5\rangle\cos t + \frac{41}{14}\langle 4, 5\rangle\sin t. \quad \square$$

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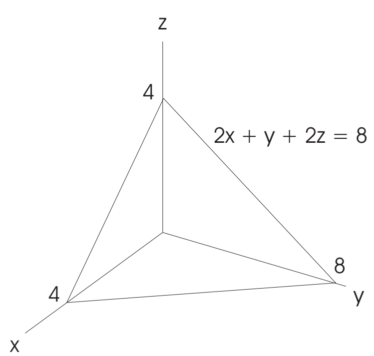
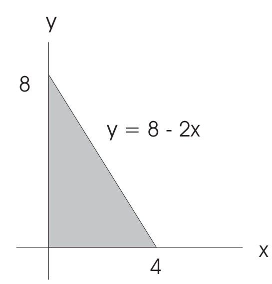
12. Find the volume of the region in the first octant cut off by the plane $2x + y + 2z = 8$.





The first picture shows the plane. The projection of the region into the x-y-plane is shown in the second picture. The projection is

$$\begin{array}{c} 0 \le x \le 4 \\ 0 \le y \le -2x + 8 \end{array}.$$

Therefore, the volume is:

$$\int_0^4 \int_0^{-2x+8} \left(4 - x - \frac{1}{2}y\right)\,dy\,dx = \int_0^4 \left[4y - xy - \frac{1}{4}y^2\right]_0^{-2x+8} dx =$$
$$\int_0^4 (x-4)^2\,dx = \left[\frac{1}{3}(x-4)^3\right]_0^4 = \frac{64}{3}. \quad \square$$

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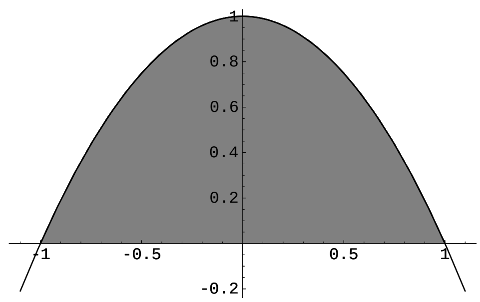
13. Compute $\displaystyle\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} e^{3x-x^3}\,dx\,dy$.

Interchange the order of integration:

$$\left\{ \begin{array}{c} 0 \le y \le 1 \\ -\sqrt{1-y} \le x \le \sqrt{1-y} \end{array} \right\} \quad \to$$



$$\to \quad \left\{ \begin{array}{c} -1 \le x \le 1 \\ 0 \le y \le 1 - x^2 \end{array} \right\}$$

Thus,

$$\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} e^{3x-x^3}\,dx\,dy = \int_{-1}^1 \int_0^{1-x^2} e^{3x-x^3}\,dy\,dx = \int_{-1}^1 (1-x^2)e^{3x-x^3}\,dx = \int_0^2 (1-x)^2 e^u \cdot \frac{du}{3(1-x^2)} =$$

$$\left[u = 3x - x^3, \quad du = (3-3x^2)\,dx = 3(1-x^2)\,dx, \quad dx = \frac{du}{3(1-x)^2}; \quad x = 0, \quad u = 0; \quad x = 1, \quad u = 2 \right]$$

$$\frac{1}{3}\int_0^2 e^u\,du = \frac{1}{3}\left[e^u\right]_0^2 = \frac{1}{3}(e^2 - 1). \quad \square$$

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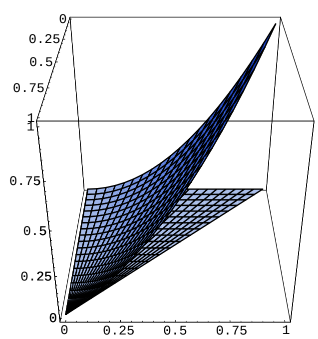
14. Compute $\displaystyle\iiint_R (6x+4y)\,dV$, where R is the region in the first octant bounded above by $z = y^2$ and bounded on the side by $x + y = 1$.



The projection of the region into the x-y-plane (the base of the solid) is the triangle given by the inequalities

$$\begin{array}{c} 0 \le x \le 1 \\ 0 \le y \le 1 - x \end{array}.$$

The top of the region is the parabolic cylinder $z = y^2$. The base of the region is the x-y plane $z = 0$. Thus, the region R is described by the inequalities is given by the inequalities

$$\begin{array}{c} 0 \le x \le 1 \\ 0 \le y \le 1 - x \\ 0 \le z \le y^2 \end{array}.$$

Therefore,

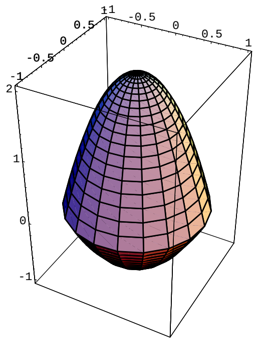
$$\iiint_R (6x+4y)\,dV = \int_0^1 \int_0^{1-x} \int_0^{y^2} (6x+4y)\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} \left[(6x+4y)z\right]_0^{y^2}\,dy\,dx =$$

$$\int_0^1 \int_0^{1-x} (6xy^2 + 4y^3)\, dy\, dx = \int_0^1 \left[2xy^3 + y^4\right]_0^{1-x} dx = \int_0^1 (1 - 2x + 2x^3 - x^4)\, dx =$$

$$\left[x - x^2 + \frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^1 = \frac{3}{10}. \quad \square$$

---

15. The solid bounded above by $z = 2 - 2x^2 - 2y^2$ and below by $z = x^2 + y^2 - 1$ has density $\rho = 2$. Find the mass and the center of mass.



By symmetry, the center of mass must lie on the z-axis, so $\overline{x} = \overline{y} = 0$.
Find the intersection of the surfaces:

$$2 - 2x^2 - 2y^2 = x^2 + y^2 - 1, \quad 3x^2 + 3y^2 = 3, \quad x^2 + y^2 = 1.$$

Thus, the projection of the region into the x-y plane is the interior of the unit circle $x^2 + y^2 = 1$. I'll convert to cylindrical. The region is

$$\begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ r^2 - 1 \le z \le 2 - 2r^2 \end{array}.$$

The mass is

$$\int_0^{2\pi} \int_0^1 \int_{r^2-1}^{2-2r^2} 2r\, dz\, dr\, d\theta = 4\pi \int_0^1 r\, [z]_{r^2-1}^{2-2r^2}\, dr = 4\pi \int_0^1 r(3 - 3r^2)\, dr = 4\pi \int_0^1 (3r - 3r^3)\, dr =$$

$$4\pi \left[\frac{3}{2}r^2 - \frac{3}{4}r^4\right]_0^1 = 3\pi.$$

The moment in the z-direction is

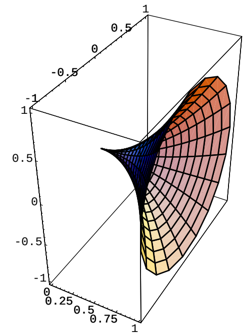
$$\int_0^{2\pi} \int_0^1 \int_{r^2-1}^{2-2r^2} 2zr\, dz\, dr\, d\theta = 4\pi \int_0^1 r \left[\frac{1}{2}z^2\right]_{r^2-1}^{2-2r^2} dr = 2\pi \int_0^1 r\left((2 - 2r^2)^2 - (r^2 - 1)^2\right) dr =$$

$$2\pi \int_0^1 \left(3r - 6r^3 + 3r^5\right) dr = 2\pi \left[\frac{3}{2}r^2 - \frac{3}{2}r^4 + \frac{1}{2}r^6\right]_0^1 = \pi.$$

Therefore, $\overline{z} = \dfrac{\pi}{3\pi} = \dfrac{1}{3}. \quad \square$

---

16. (a) Parametrize the surface generated by revolving $y = x^2$, $0 \le x \le 1$, about the x-axis.



$$x = u, \quad y = u^2 \cos v, \quad z = u^2 \sin v, \qquad 0 \le u \le 1, \quad 0 \le v \le 2\pi. \quad \square$$

(b) Find the area of the surface.

You may want to make use of the following formula:

$$\int u^2 \sqrt{a^2u^2+1}\, du = \frac{1}{a^3}\left(\frac{au}{4}(a^2u^2+1)^{3/2} - \frac{au}{8}\sqrt{a^2u^2+1} - \frac{1}{8}\ln|\sqrt{a^2u^2+1}+au|\right) + C.$$

The normal vector is

$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2u\cos v & 2u \sin v \\ 0 & -u^2 \sin v & -u^2 \cos v \end{vmatrix} = \langle 2u^3, -u^2 \cos v, -u^2 \sin v\rangle.$$

The length of the normal is

$$\|\overrightarrow{T_u} \times \overrightarrow{T_v}\| = \left(4u^6 + u^4(\cos v)^2 + u^4(\sin v)^2\right)^{1/2} = u^2\sqrt{4u^2+1}.$$

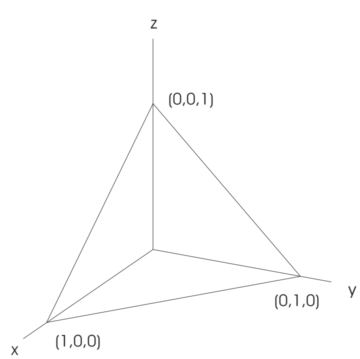
Hence, the area is

$$\int_0^{2\pi}\int_0^1 u^2\sqrt{4u^2+1}\, du\, dv = 2\pi\left[\frac{u}{16}(4u^2+1)^{3/2} - \frac{u}{32}\sqrt{4u^2+1} - \frac{1}{64}\ln|\sqrt{4u^2+1}+2u|\right]_0^1 =$$

$$\pi\left(\frac{1}{8}5^{3/2} - \frac{1}{16}\sqrt{5} - \frac{1}{32}\ln(\sqrt{5}+2)\right) \approx 3.80973. \quad \square$$

---

17. A wire is made of the three segments connecting the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. The density of the wire is $\delta = x+y+z$. Find its mass.



By symmetry, the mass is three times the mass of one of the segments. I will use the segment in the x-y plane: It is the part of the line $x+y=1$ which goes from $x=0$ to $x=1$.

I can parametrize the line by setting $x=t$ with $0 \le t \le 1$, so $y=1-t$. This is in the x-y plane, so $z=0$. Then

$$\vec{\sigma}\,'(t) = \langle 1, -1 \rangle, \quad \text{so} \quad \|\vec{\sigma}\,'(t)\| = \sqrt{2}.$$

Since $\delta(t) = t + (1-t) + 0 = 1$, the path integral for this segment is

$$\int_{\vec{\sigma}} \delta\, ds = \int_0^1 1 \cdot \sqrt{2}\, dt = \sqrt{2}.$$

The mass of the whole wire is $3\sqrt{2} \approx 4.24264$. $\square$

---

18. Let

$$\vec{\sigma}(t) = \left\langle te^{t-1}, t^3, \sin\left(\frac{\pi t}{2}\right)\right\rangle, \quad 0 \le t \le 1.$$

Compute

$$\int_{\vec{\sigma}} (y - z^2)\, dx + (x - 2y + 2yz)\, dy + (y^2 + 2z - 2xz)\, dz.$$

Let $\vec{F} = \langle y - z^2, x - 2y + 2yz, y^2 + 2z - 2xz \rangle$. Then

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y - z^2 & x - 2y + 2yz & y^2 + 2z - 2xz \end{vmatrix} = \langle 0, 0, 0 \rangle.$$

Hence, the field is conservative. A potential function f must satisfy

$$\frac{\partial f}{\partial x} = y - z^2, \quad \frac{\partial f}{\partial y} = x - 2y + 2yz, \quad \frac{\partial f}{\partial z} = y^2 + 2z - 2xz.$$

Integrate the first equation with respect to x:

$$f = \int (y - z^2)\, dx = yx - xz^2 + C(y, z).$$

Since the integral is with respect to x, the arbitrary constant may depend on y and z. Differentiate with respect to y:

$$x - 2y + 2yz = \frac{\partial f}{\partial y} = x + \frac{\partial C}{\partial y}.$$

$\dfrac{\partial C}{\partial y} = -2y + 2yz$, so

$$C = \int (-2y + 2yz)\, dy = -y^2 + y^2 z + D(z).$$

Since the integral is with respect to y, the arbitrary constant may depend on z. Now

$$f = yx - xz^2 - y^2 + y^2 z + D(z).$$

Differentiate with respect to z:

$$y^2 + 2z - 2xz = \frac{\partial f}{\partial z} = -2xz + y^2 + \frac{dD}{dz}.$$

$\dfrac{dD}{dz} = 2z$, so $D = z^2 + E$. At this point, E is a numerical constant; since the derivative of a number is 0, and since I only need *some* potential function, I may take $E = 0$. Then $D = z^2$, so

$$f = yx - xz^2 - y^2 + y^2 z + z^2.$$

Now use path independence. The endpoints of the path are

$$\vec{\sigma}(0) = \langle 0,0,0\rangle, \quad \vec{\sigma}(1) = \langle 1,1,1\rangle.$$

Hence,

$$\int_{\vec{\sigma}} (y - z^2)\,dx + (x - 2y + 2yz)\,dy + (y^2 + 2z - 2xz)\,dz = f(1,1,1) - f(0,0,0) = 1. \quad \square$$

---

19. Let

$$\vec{F} = \left\langle \frac{x}{x^2+y^2+z^2+1}, \frac{y}{x^2+y^2+z^2+1}, \frac{z}{x^2+y^2+z^2+1} \right\rangle,$$

and let $\vec{\sigma}(t)$ be *any* path from *any* point on the sphere $x^2+y^2+z^2 = 1$ to *any* point on the sphere $x^2+y^2+z^2 = 5$. Compute $\displaystyle\int_{\vec{\sigma}} \vec{F}\cdot d\vec{s}$.

Let P be a point on the sphere $x^2+y^2+z^2 = 1$ and let Q be a point on the sphere $x^2+y^2+z^2 = 5$. If $f(x,y,z) = \dfrac{1}{2}\ln(x^2+y^2+z^2+1)$, then

$$\nabla f = \left\langle \frac{x}{x^2+y^2+z^2+1}, \frac{y}{x^2+y^2+z^2+1}, \frac{z}{x^2+y^2+z^2+1} \right\rangle.$$

Hence, by path independence,

$$\int_{\vec{\sigma}} \vec{F}\cdot d\vec{s} = f(Q) - f(P) = \frac{1}{2}\ln(5+1) - \frac{1}{2}\ln(1+1) = \frac{1}{2}(\ln 6 - \ln 2) \approx 0.54931.$$

Explanation: Since Q is on $x^2+y^2+z^2 = 5$, for this point $\ln(x^2+y^2+z^2+1) = \ln(5+1)$. Likewise, $\ln(x^2+y^2+z^2+1) = \ln(1+1)$ for P, because P is on $x^2+y^2+z^2 = 1$. $\square$

---

20. Use Green's Theorem to show that the area of the ellipse

$$x = a\cos t, \quad y = b\sin t, \qquad 0 \le t \le 2\pi$$

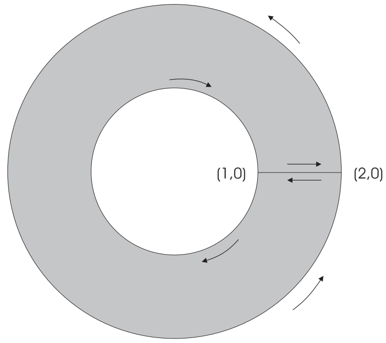
is πab.

$\dfrac{dy}{dt} = b\cos t$, so the area is

$$\int_C x\,dy = \int_0^{2\pi} ab(\cos t)^2\,dt = ab\int_0^{2\pi} \frac{1}{2}(1+\cos 2t)\,dt = ab\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = \pi ab. \quad \square$$

---

21. Let $\vec{\sigma}$ be the path which starts at $(2,0)$, goes around the circle $x^2+y^2=4$ in the counterclockwise direction, traverses the segment from $(2,0)$ to $(1,0)$, goes around the circle $x^2+y^2=1$ in the clockwise direction, and traverses the segment from $(1,0)$ to $(2,0)$. Compute $\displaystyle\int_{\vec{\sigma}} -y\,dx + x\,dy$.



Let R denote the ring-shaped area between the two circles. By Green's theorem,

$$\int_{\vec{\sigma}} -y\,dx + x\,dy = \iint_R (1-(-1))\,dx\,dy = 2\cdot(\text{area of } R).$$

The area of R is the area of the outer circle minus the area of the inner circle, or $4\pi - \pi = 3\pi$. Hence,

$$\int_{\vec{\sigma}} -y\,dx + x\,dy = 6\pi \approx 18.84956.\quad\square$$

---

22. Compute the circulation of $\vec{F} = \langle yz, xz, -xy\rangle$ counterclockwise (as viewed from above) around the triangle with vertices $A(1,2,1)$, $B(2,1,4)$, and $C(-3,1,1)$.

$\overrightarrow{AB} = \langle 1,-1,3\rangle$ and $\overrightarrow{AC} = \langle -4,-1,0\rangle$, so the triangle may be parametrized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -1 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

The limits are

$$0 \le u \le 1, \quad 0 \le v \le 1-u.$$

Reason: If I use $0 \le u \le 1$, $0 \le v \le 1$, the input is a square and the output is a parallelogram (four-sided figure to four-sided figure). Since I only want a triangle — half the parallelogram — I only feed half the square into the transformation (three-sided figure to three-sided figure).

In component form, this is

$$x = u - 4v + 1, \quad y = -u - v + 2, \quad z = 3u + 1.$$

The normal is

$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 3 \\ -4 & -1 & 0 \end{vmatrix} = \langle 3, -12, -5\rangle.$$

I need the upward normal (the boundary is traversed "counterclockwise as viewed from above"), so I negate this vector to get $\langle -3, 12, 5\rangle$.

Now

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & -xy \end{vmatrix} = \langle -2x, 2y, 0\rangle.$$

So

$$\operatorname{curl}\vec{F}\cdot(\vec{T_u}\times\vec{T_v}) = 54 - 18u - 48v.$$

By Stokes' theorem, the circulation of $\vec{F}$ around the boundary of the triangle is

$$\int_0^1\int_0^{1-u}(54-18u-48v)\,dv\,du = \int_0^1 \left[(54-18u)v - 24v^2\right]_0^{1-u}\,du =$$

$$\int_0^1 \left((54-18u)(1-u) - 24(1-u)^2\right)\,du = \int_0^1 \left(54 - 72u + 18u^2 - 24(1-u)^2\right)\,du =$$

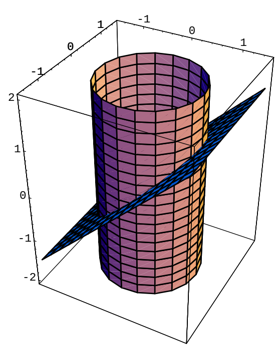
$$\left[54u - 36u^2 + 6u^3 + 8(1-u)^3\right]_0^1 = 16. \quad \square$$

---

23. Let $\vec{\sigma}$ be the curve of intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$, traversed counterclockwise as viewed from above. Compute the circulation of $\vec{F} = \langle x^2y^3, 1, z\rangle$ around $\vec{\sigma}$:

(a) Directly, by parametrizing the curve and computing the line integral.

(b) Using Stokes' theorem.



First, I'll compute the circulation directly. To parametrize the curve of intersection, project it into the x-y plane. I get $x^2 + y^2 = 1$, which I can parametrize by $x = \cos t$, $y = \sin t$. Now $z = x$, so $z = \cos t$. Thus, the curve is

$$\vec{\sigma}(t) = \langle \cos t, \sin t, \cos t\rangle, \quad 0 \le t \le 2\pi.$$

Now

$$\vec{F}(t) = \langle (\cos t)^2(\sin t)^3, 1, \cos t\rangle,$$

$$\vec{\sigma}\,'(t) = \langle -\sin t, \cos t, -\sin t\rangle,$$

$$\vec{F}(t)\cdot\vec{\sigma}\,'(t) = -(\cos t)^2(\sin t)^4 + \cos t - \sin t\cos t.$$

The circulation is

$$\int_0^{2\pi}\left(-(\cos t)^2(\sin t)^4 + \cos t - \sin t\cos t\right)\,dt = -\frac{\pi}{8}.$$

Note that you can integrate $(\cos t)^2(\sin t)^4$ by using the double angle formulas, but it's a little messy.

Next, I'll use Stokes' theorem. The surface is the plane $z = x$, for which the normal is

$$\vec{N} = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle = \langle -1, 0, 1 \rangle.$$

The z-component is positive, so this is the upward normal, consistent with the orientation of the curve. Next,

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = \langle 0, 0, -3x^2y^2 \rangle.$$

Hence,

$$\operatorname{curl} \vec{F} \cdot \vec{N} = -3x^2y^2.$$

The projection of the surface into the x-y plane is the interior of the unit circle. I'll convert to polar. The projection is

$$\begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix}.$$

Moreover,

$$\operatorname{curl} \vec{N} \cdot \vec{N} = -3r^4 (\cos\theta)^2 (\sin\theta)^2.$$

So by Stokes' theorem, the circulation is

$$\int_0^{2\pi} \int_0^1 -3r^4 (\cos\theta)^2 (\sin\theta)^2 r \, dr \, d\theta = \int_0^{2\pi} (\cos\theta)^2 (\sin\theta)^2 \left[-\frac{1}{2} r^6 \right]_0^1 d\theta =$$

$$-\frac{1}{2} \int_0^{2\pi} (\cos\theta)^2 (\sin\theta)^2 \, d\theta = -\frac{1}{8} \int_0^{2\pi} (1 + \cos 2\theta)(1 - \cos 2\theta) \, d\theta = -\frac{1}{8} \int_0^{2\pi} (1 - (\cos 2\theta)^2) \, d\theta =$$

$$-\frac{1}{8} \int_0^{2\pi} (\sin 2\theta)^2 \, d\theta = -\frac{1}{16} \int_0^{2\pi} (1 - \cos 4\theta) \, d\theta = -\frac{1}{16} \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{2\pi} = -\frac{\pi}{8}. \quad \square$$

---

24. Let R be the solid region in the first octant cut off by the sphere $x^2 + y^2 + z^2 = 1$. Compute the flux out through the boundary of R of the vector field

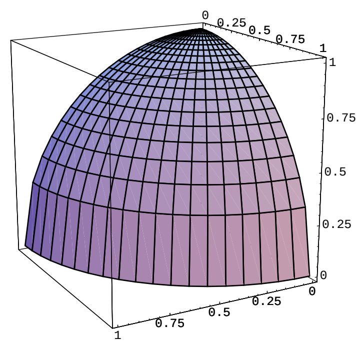
$$\vec{F} = \left\langle x^3 + yz, y^3 - xz, z^3 + 2xy \right\rangle.$$

By the Divergence Theorem,

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div} \vec{F} \, dV.$$

I'll use spherical coordinates.

$$\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 3z^2 = 3\rho^2.$$



The region is

$$\begin{array}{c} 0 \le \theta \le \dfrac{\pi}{2} \\ 0 \le \rho \le 1 \\ 0 \le \phi \le \dfrac{\pi}{2} \end{array}.$$

Therefore,

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \int_0^{\pi/2} \int_0^1 \int_0^{\pi/2} 3\rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{3\pi}{2} \int_0^1 \rho^4 \left[-\cos\phi\right]_0^{\pi/2} \, d\rho = \frac{3\pi}{2} \int_0^1 \rho^4 \, d\rho =$$

$$\frac{3\pi}{2} \left[\frac{1}{5}\rho^5\right]_0^1 = \frac{3\pi}{10}. \quad \square$$

The best thing for being sad is to learn something. - MERLYN, in T. H. WHITE'S *The Once and Future King*