Summary of Convergence Tests for Series

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series of **positive** terms.

The series $\sum_{n=1}^{\infty} a_n$ converges if and only if the *sequence* of partial sums, $S_n = a_1 + a_2 + a_3 + \cdots a_n$, converges.

NOTE: $\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$

Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example: The series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$ is divergent since $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}} = 1$

This means that the **terms** of a convergent series must approach zero. That is, if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. **However,** $\lim_{n\to\infty} a_n = 0$ **does not imply convergence.**

Geometric Series: THIS is our model series A geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges for $-1 < r < 1$.

Note: $r = \frac{a_{n+1}}{a_n}$ If the series converges, the sum of the series is $\frac{a}{1-r}$.

Example: The series $\sum_{n=1}^{\infty} 5\left(\frac{7}{8}\right)^n$ converges with $a = a_1 = \frac{35}{8}$ and $r = \frac{7}{8}$. The sum of the series is 35.

Integral Test: If *f* is a continuous, positive, decreasing function on $[1,\infty)$ with $f(n) = a_n$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges.

***p*-series:** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and **diverges otherwise**.

Examples: The series $\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$ is convergent but the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Ratio Test: (a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges; (b) if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ the series *diverges*. **Otherwise**, you must use a different test for convergence.

This says that if the series eventually behaves like a convergent (divergent) geometric series, it converges (diverges). If this limit is **one**, the test is inconclusive and a different test is required. Specifically**, the Ratio Test <u>does not</u> work for *p*-series.**

Comparison Test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(a) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ diverges.

The Comparison Test requires that you make one of two comparisons:

- Compare an unknown series to a LARGER known *convergent* series **(smaller than convergent is convergent)**
- Compare an unknown series to a SMALLER known *divergent* series (**bigger than divergent is divergent**)

Examples: $\sum_{n=2}^{\infty} \frac{3n}{n^2-2} > \sum_{n=2}^{\infty} \frac{3n}{n^2} = 3\sum_{n=2}^{\infty} \frac{1}{n}$ which is a *divergent* harmonic series. Since the original series is larger by comparison, it is divergent.

We have $\sum_{n=1}^{\infty} \frac{5n}{2n^3+n^2+1} < \sum_{n=1}^{\infty} \frac{5n}{2n^3} = \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series. Since the original series is smaller by comparison, it is convergent.

Limit Comparison Test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If

$\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $0 < c < \infty$, then either both series converge or both series diverge. **(Useful for *p*-series)**

Rule of Thumb: To obtain a series for comparison, omit lower order terms in the numerator and the denominator and then simplify.

Examples: For the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+3}$, compare to $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent p-series.

For the series $\sum_{n=1}^{\infty} \frac{\pi^n + \sqrt{n}}{3^n + n^2}$, compare to $\sum_{n=1}^{\infty} \frac{\pi^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{\pi}{3}\right)^n$ which is a divergent geometric series.

Alternating Series Test: If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$

satisfies (a) $b_n > b_{n+1}$ and (b) $\lim_{n\to\infty} b_n = 0$, then the series converges.

Remainder: $|R_n| = |s - s_n| \le b_{n+1}$

Absolute convergence simply means that the series converges *without* alternating (all signs and terms are positive).

Examples: The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is convergent but *not* absolutely convergent.

Alternating *p*-series: The alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges for $p > 0$.

Examples: The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ and the Alternating Harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ are convergent.