

# Differential Equations Study Guide<sup>1</sup>

## First Order Equations

(1) **General Form of ODE:**  $\frac{dy}{dx} = f(x, y)$

(2) **Initial Value Problem:**  $y' = f(x, y), y(x_0) = y_0$

### Linear Equations

(3) **General Form:**  $y' + p(x)y = f(x)$

(4) **Integrating Factor:**  $\mu(x) = e^{\int p(x)dx}$

(5)  $\implies \frac{d}{dx}(\mu(x)y) = \mu(x)f(x)$

(6) **General Solution:**  $y = \frac{1}{\mu(x)} \left( \int \mu(x)f(x)dx + C \right)$

### Homogeneous Equations

(7) **General Form:**  $y' = f(y/x)$

(8) **Substitution:**  $y = zx$

(9)  $\implies y' = z + xz'$

The result is always separable in  $z$ :

(10)  $\frac{dz}{f(z) - z} = \frac{dx}{x}$

### Bernoulli Equations

(11) **General Form:**  $y' + p(x)y = q(x)y^n$

(12) **Substitution:**  $z = y^{1-n}$

The result is always linear in  $z$ :

(13)  $z' + (1-n)p(x)z = (1-n)q(x)$

### Exact Equations

(14) **General Form:**  $M(x, y)dx + N(x, y)dy = 0$

(15) **Text for Exactness:**  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(16) **Solution:**  $\phi = C$  where

(17)  $M = \frac{\partial \phi}{\partial x}$  and  $N = \frac{\partial \phi}{\partial y}$

### Method for Solving Exact Equations:

1. Let  $\phi = \int M(x, y)dx + h(y)$

2. Set  $\frac{\partial \phi}{\partial y} = N(x, y)$

3. Simplify and solve for  $h(y)$ .

4. Substitute the result for  $h(y)$  in the expression for  $\phi$  from step 1 and then set  $\phi = 0$ . This is the solution.

Alternatively:

1. Let  $\phi = \int N(x, y)dy + g(x)$

2. Set  $\frac{\partial \phi}{\partial x} = M(x, y)$

3. Simplify and solve for  $g(x)$ .

4. Substitute the result for  $g(x)$  in the expression for  $\phi$  from step 1 and then set  $\phi = 0$ . This is the solution.

### Integrating Factors

**Case 1:** If  $P(x, y)$  depends only on  $x$ , where

(18)  $P(x, y) = \frac{M_y - N_x}{N} \implies \mu(y) = e^{\int P(x)dx}$

then

(19)  $\mu(x)M(x, y)dx + \mu(x)N(x, y)dy = 0$

is exact.

**Case 2:** If  $Q(x, y)$  depends only on  $y$ , where

(20)  $Q(x, y) = \frac{N_x - M_y}{M} \implies \mu(y) = e^{\int Q(y)dy}$

Then

(21)  $\mu(y)M(x, y)dx + \mu(y)N(x, y)dy = 0$

is exact.

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# Second Order Linear Equations

## General Form of the Equation

(22) **General Form:**  $a(t)y'' + b(t)y' + c(t)y = g(t)$

(23) **Homogeneous:**  $a(t)y'' + b(t)y' + c(t)y = 0$

(24) **Standard Form:**  $y'' + p(t)y' + q(t)y = f(t)$

The **general solution** of (22) or (24) is

(25) 
$$y = C_1y_1(t) + C_2y_2(t) + y_p(t)$$

where  $y_1(t)$  and  $y_2(t)$  are linearly independent solutions of (23).

## Linear Independence and The Wronskian

Two functions  $f(x)$  and  $g(x)$  are **linearly dependent** if there exist numbers  $a$  and  $b$ , not both zero, such that  $af(x) + bg(x) = 0$  for all  $x$ . If no such numbers exist then they are **linearly independent**.

If  $y_1$  and  $y_2$  are two solutions of (23) then

(26) **Wronskian:**  $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

(27) **Abel's Formula:**  $W(t) = Ce^{-\int p(t)dt}$

and the following are all equivalent:

1.  $\{y_1, y_2\}$  are linearly independent.
2.  $\{y_1, y_2\}$  are a fundamental set of solutions.
3.  $W(y_1, y_2)(t_0) \neq 0$  at some point  $t_0$ .
4.  $W(y_1, y_2)(t) \neq 0$  for all  $t$ .

## Initial Value Problem

(28) 
$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases}$$

## Linear Equation: Constant Coefficients

(29) **Homogeneous:**  $ay'' + by' + cy = 0$

(30) **Non-homogeneous:**  $ay'' + by' + cy = g(t)$

(31) **Characteristic Equation:**  $ar^2 + br + c = 0$

(32) **Quadratic Roots:**  $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The solution of (29) is given by:

(33) **Real Roots** ( $r_1 \neq r_2$ ):  $y_h = C_1e^{r_1t} + C_2e^{r_2t}$

(34) **Repeated** ( $r_1 = r_2$ ):  $y_h = (C_1 + C_2t)e^{r_1t}$

(35) **Complex** ( $r = \alpha \pm i\beta$ ):  $y_H = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$

The solution of (30) is  $y = y_p + y_h$  where  $y_h$  is given by (33) through (35) and  $y_p$  is found by **undetermined coefficients** or **reduction of order**.

## Heuristics for Undetermined Coefficients (Trial and Error)

If $f(t) =$	then guess that a particular solution $y_p =$
$P_n(t)$	$t^s(A_0 + A_1t + \dots + A_nt^n)$
$P_n(t)e^{at}$	$t^s(A_0 + A_1t + \dots + A_nt^n)e^{at}$
$P_n(t)e^{at} \sin bt$ or $P_n(t)e^{at} \cos bt$	$t^s e^{at} [(A_0 + A_1t + \dots + A_nt^n) \cos bt + (A_0 + A_1t + \dots + A_nt^n) \sin bt]$

## Method of Reduction of Order

When solving (23), given  $y_1$ , then  $y_2$  can be found by solving

(36) 
$$y_1y_2' - y_1'y_2 = Ce^{-\int p(t)dt}$$

The solution is given by

(37) 
$$y_2 = y_1 \int \frac{e^{-\int p(x)dx} dx}{y_1(x)^2}$$

## Method of Variation of Parameters

If  $y_1(t)$  and  $y_2(t)$  are a fundamental set of solutions to (23) then a particular solution to (24) is

(38) 
$$y_P(t) = -y_1(t) \int \frac{y_2(t)f(t)}{W(t)} dt + y_2(t) \int \frac{y_1(t)f(t)}{W(t)} dt$$

## Cauchy-Euler Equation

(39) **ODE:**  $ax^2y'' + bxy' + cy = 0$

(40) **Auxilliary Equation:**  $ar(r-1) + br + c = 0$

The solutions of (39) depend on the roots  $r_{1,2}$  of (40):

(41) **Real Roots:**  $y = C_1x^{r_1} + C_2x^{r_2}$

(42) **Repeated Root:**  $y = C_1x^r + C_2x^r \ln x$

(43) **Complex:**  $y = x^\alpha [C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)]$

In (43)  $r_{1,2} = \alpha \pm i\beta$ , where  $\alpha, \beta \in \mathbb{R}$

## Series Solutions

(44)  $(x - x_0)^2y'' + (x - x_0)p(x)y' + q(x)y = 0$

If  $x_0$  is a **regular point** of (44) then

(45) 
$$y_1(t) = (x - x_0)^n \sum_{k=0}^{\infty} a_k(x - x_k)^k$$

At a **Regular Singular Point**  $x_0$ :

(46) **Indicial Equation:**  $r^2 + (p(0) - 1)r + q(0) = 0$

(47) **First Solution:**  $y_1 = (x - x_0)^{r_1} \sum_{k=0}^{\infty} a_k(x - x_k)^k$

Where  $r_1$  is the larger real root if both roots of (46) are real or either root if the solutions are complex.