Differential Equations Study Guide[1]

First Order Equations

(1) **General Form of ODE:** $\dfrac{dy}{dx} = f(x,y)$

(2) **Initial Value Problem:** $y' = f(x,y),\ y(x_0) = y_0$

Linear Equations

(3) **General Form:** $y' + p(x)y = f(x)$

(4) **Integrating Factor:** $\mu(x) = e^{\int p(x)dx}$

(5) $\implies \dfrac{d}{dx}(\mu(x)y) = \mu(x)f(x)$

(6) **General Solution:** $y = \dfrac{1}{\mu(x)}\left(\int \mu(x)f(x)dx + C\right)$

Homogeneous Equations

(7) **General Form:** $y' = f(y/x)$

(8) **Substitution:** $y = zx$

(9) $\implies y' = z + xz'$

The result is always separable in z:

(10) $\dfrac{dz}{f(z) - z} = \dfrac{dx}{x}$

Bernoulli Equations

(11) **General Form:** $y' + p(x)y = q(x)y^n$

(12) **Substitution:** $z = y^{1-n}$

The result is always linear in z:

(13) $z' + (1-n)p(x)z = (1-n)q(x)$

Exact Equations

(14) **General Form:** $M(x,y)dx + N(x,y)dy = 0$

(15) **Text for Exactness:** $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

(16) **Solution:** $\phi = C$ where

(17) $M = \dfrac{\partial \phi}{\partial x}$ and $N = \dfrac{\partial \phi}{\partial y}$

Method for Solving Exact Equations:

1. Let $\phi = \int M(x,y)dx + h(y)$
2. Set $\dfrac{\partial \phi}{\partial y} = N(x,y)$
3. Simplify and solve for $h(y)$.
4. Substitute the result for $h(y)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Alternatively:

1. Let $\phi = \int N(x,y)dy + g(x)$
2. Set $\dfrac{\partial \phi}{\partial x} = M(x,y)$
3. Simplify and solve for $g(x)$.
4. Substitute the result for $g(x)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

Integrating Factors

Case 1: If $P(x,y)$ depends only on x, where

(18) $P(x,y) = \dfrac{M_y - N_x}{N} \implies \mu(y) = e^{\int P(x)dx}$

then

(19) $\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$

is exact.

Case 2: If $Q(x,y)$ depends only on y, where

(20) $Q(x,y) = \dfrac{N_x - M_y}{M} \implies \mu(y) = e^{\int Q(y)dy}$

Then

(21) $\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$

is exact.

Second Order Linear Equations

General Form of the Equation

(22) **General Form:** $a(t)y'' + b(t)y' + c(t)y = g(t)$

(23) **Homogeneous:** $a(t)y'' + b(t)y' + c(t)y = 0$

(24) **Standard Form:** $y'' + p(t)y' + q(t)y = f(t)$

The **general solution** of (22) or (24) is

(25) $$y = C_1y_1(t) + C_2y_2(t) + y_p(t)$$

where $y_1(t)$ and $y_2(t)$ are linearly independent solutions of (23).

Linear Independence and The Wronskian

Two functions $f(x)$ and $g(x)$ are **linearly dependent** if there exist numbers a and b, not both zero, such that $af(x) + bg(x) = 0$ for all x. If no such numbers exist then they are **linearly independent**.

If y_1 and y_2 are two solutions of (23) then

(26) **Wronskian:** $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

(27) **Abel's Formula:** $W(t) = Ce^{-\int p(t)dt}$

and the following are all equivalent:

1. $\{y_1, y_2\}$ are linearly independent.
2. $\{y_1, y_2\}$ are a fundamental set of solutions.
3. $W(y_1, y_2)(t_0) \neq 0$ at some point t_0.
4. $W(y_1, y_2)(t) \neq 0$ for all t.

Initial Value Problem

(28) $$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{cases}$$

Linear Equation: Constant Coefficients

(29) **Homogeneous:** $ay'' + by' + cy = 0$

(30) **Non-homogeneous:** $ay'' + by' + cy = g(t)$

(31) **Characteristic Equation:** $ar^2 + br + c = 0$

(32) **Quadratic Roots:** $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The solution of (29) is given by:

(33) **Real Roots**$(r_1 \neq r_2) : y_h = C_1e^{r_1t} + C_2e^{r_2t}$

(34) **Repeated**$(r_1 = r_2) : y_h = (C_1 + C_2t)e^{r_1t}$

(35) **Complex**$(r = \alpha \pm i\beta) : y_H = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$

The solution of (30) is $y = y_p + y_h$ where y_h is given by (33) through (35) and y_p is found by **undetermined coefficients** or **reduction of order**.

Heuristics for Undetermined Coefficients (Trial and Error)

If $f(t) =$	then guess that a particular solution $y_p =$
$P_n(t)$	$t^s(A_0 + A_1t + \cdots + A_nt^n)$
$P_n(t)e^{at}$	$t^s(A_0 + A_1t + \cdots + A_nt^n)e^{at}$
$P_n(t)e^{at}\sin bt$ or $P_n(t)e^{at}\cos bt$	$t^se^{at}[(A_0 + A_1t + \cdots + A_nt^n)\cos bt$ $+(A_0 + A_1t + \cdots + A_nt^n)\sin bt]$

Method of Reduction of Order

When solving (23), given y_1, then y_2 can be found by solving

(36) $$y_1y_2' - y_1'y_2 = Ce^{-\int p(t)dt}$$

The solution is given by

(37) $$y_2 = y_1 \int \frac{e^{-\int p(x)dx}dx}{y_1(x)^2}$$

Method of Variation of Parameters

If $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to (23) then a particular solution to (24) is

(38) $$y_P(t) = -y_1(t)\int \frac{y_2(t)f(t)}{W(t)}dt + y_2(t)\int \frac{y_1(t)f(t)}{W(t)}dt$$

Cauchy-Euler Equation

(39) **ODE:** $ax^2y'' + bxy' + cy = 0$

(40) **Auxilliary Equation:** $ar(r-1) + br + c = 0$

The solutions of (39) depend on the roots $r_{1,2}$ of (40):

(41) **Real Roots:** $y = C_1x^{r_1} + C_2x^{r_2}$

(42) **Repeated Root:** $y = C_1x^r + C_2x^r \ln x$

(43) **Complex:** $y = x^\alpha[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)]$

In (43) $r_{1,2} = \alpha \pm i\beta$, where $\alpha,\beta \in \mathbb{R}$

Series Solutions

(44) $$(x - x_0)^2y'' + (x - x_0)p(x)y' + q(x)y = 0$$

If x_0 is a **regular point** of (44) then

(45) $$y_1(t) = (x - x_0)^n \sum_{k=0}^{\infty} a_k(x - x_k)^k$$

At a **Regular Singular Point** x_0:

(46) **Indicial Equation:** $r^2 + (p(0) - 1)r + q(0) = 0$

(47) **First Solution:** $y_1 = (x - x_0)^{r_1} \sum_{k=0}^{\infty} a_k(x - x_k)^k$

Where r_1 is the larger real root if both roots of (46) are real or either root if the solutions are complex.